# Hands-on methods <br> for teaching Form 1 and 2 maths in Malawi 

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## 1 Introduction

The main aim of this book is to improve learners understanding of mathematical ideas. This can be achieved by using engaging teaching and learning methods (hands-on), and having learners sharing their thinking with each other (talking about what they are doing in groups).
You will notice that the content of this book is arranged differently from the syllabus. Topics that belong together are placed together - this encourages understanding as the ideas carry over from one topic to the next. Here we are not trying to develop rote memory of topics that are not understood, but developing understanding that will lead to the ability to think, and to solve problems.
Some Malawian primary teachers have had extraordinary success using hands-on methods in upper primary classes; this is reported in Prince Kumwenda and Calister Phiri's Hands-on, Minds-on. They report that learners are much more engaged and happier in maths classes when they are able to work together with their friends to learn, talk and understand what is being taught. It is known that this works in other countries at both primary and secondary levels - particularly lower secondary - and this book provides ideas of how this might be achieved at lower secondary in Malawi.
Why hands-on methods?
'Hands-on' means actually working with simple equipment using hands - and minds. The equipment needed varies between topics; the choice of equipment is made to match the mathematical concepts that we want the learners to understand.

## Learners working in groups

Research shows that most of the talk in maths classes is by the teacher. Yet it is known that when learners' minds are engaged in understanding a topic they need to talk to others about their ideas. In this way they develop their understandings, and help each other to build self-confidence.

## Games in groups

In the words of one Malawian teacher who uses games: "games add value and variety in the teaching process and they also make students appreciate and have wide range of understanding of the concepts." Many games may be played by a small group using a pack of playing cards, making use of the numbers to generate much practice in an enjoyable way. Equipment
Much of the equipment for the activities below is paper. There are some templates for you to use at the $b$ ack of this book. The paper will often be cut, so you will need many pairs of scissors. In addition you should start collecting bottle tops.
How to introduce each hands-on activity

- Make it for yourself and make sure you can do it and you understand.
- Explain to the class that you are going to demonstrate and then they will practise.
- Demonstrate the activity, and explain how it relates to the mathematical ideas.
- Give out equipment to each group, and give them instructions.
- At the end, collect the equipment for later use.

Using spreadsheets
To reinforce the group learning and the textbook practice, there are many spreadsheets provided. Most of these use diagrams, and so assist students to develop their understanding of the ideas as well as practising the skills.

## Getting started

Try something that seems simple first. And even if it doesn't work well the first time, don't give up. It might take you a few attempts before you feel that you are making the most of hands-on activities to improve your students' learning.
Learners' attitudes to hands-on group learning
You may find that learners initial resist the idea of getting into groups and working things out for themselves. It is partly because it is a new experience, and partly because they have to 'think', often for the first time, about what mathematical ideas are about.
The time it takes
Teachers of mathematics who try these group, hands-on activities for the first time often point out that it takes longer to get through the topics than when they just tell the students what to do. This is true; it does take longer, but the greatly improved results will justify the time. Because the ideas will be better understood by students, you should be able to save time with fewer practice exercises, and less review teaching.

## 2 Primary revision

## Introduction

Many secondary learners have learned the maths that they know by rote. Many do not understand it. There is some overlap between the primary and secondary syllabuses, but some topics are not revised, although it is vital that they are mastered for future success. So it seems that some time should be spent making sure these topics are properly understood.

## a Base 10 and place value for whole numbers and decimals

1 Base 10 sheets to show place value for whole numbers
We can use paper to show the sizes of the digits for place value. It shows that each ten is the same area $s$ ten 1 s , and each hundred is the same area as 10 tens.
We can use the same paper squares and strips to show decimals. Call the big square 100. Then the strips are tens, and the little squares are ones.


One hundred


You may photocopy the template on pages $x x$ and $x x$ to make these pieces available for your class.
Students should be asked to show various numbers, including some with zeros, such as 304 . 2 Base 10 sheets to show place value for decimals
We can use the same paper squares and strips to show decimals. Call the big square 1.
Then the strips are tenths, and the little squares are hundredths.


Students should be asked to show various numbers, including some with zeros, such as 3.04. 3 Tape measures or rulers
On a tape measure we might measure lengths in metres. So the length of an A4 sheet is about 0.3 m , or more exactly 0.297 m . In centimetres this is about 30 cm , or more exactly 29.7 cm . students should measure lengths over 1 m , and over 10 m , and convert them to centimetres as well.
4 Multiply and divide by 10
As you divide by 10,100 or 1,000 you see the digits move to the right.

$$
\begin{array}{r}
123 . \mathrm{mm} \\
12.3 \mathrm{~cm} \\
0.123 \mathrm{~m}
\end{array}
$$

A digit slider shows the same thing. Make one out of a strip of paper showing digits widely separated, so a fixed decimal point can be used to represent the place value. The point separates the whole numbers from the decimal fractions.


Divide 123 by 10


Divide 12.3 by 100 . Note the new 0 .


Please note: when we divide a number by 10 (or 100 etc.) the digits move right, but THE DECIMAL POINT DOES NOT MOVE.

## 5 Multiply two decimals

Students should draw a $10 \times 10$ grid, showing an area of 1 . Its width and height are both lengths of 1 . They recognise that there are 100 little squares, so each little square shows one hundredth (0.01).
They may then shade an area of, for example, $0.2 \times 0.3$. This will cover 2 squares one way and 3 squares the other, and so the area $0.2 \times 0.3$ is 6 hundredths or 0.06 .
This may then be extended to rectangles that stretch outside the single square, such as $0.5 \times 1.4$ and $1.6 \times 1.8$ shown.
In each case the number of hundredths is just the product of the whole numbers.


## Spreadsheets

## Odometer

This show a demonstration odometer, such as in a car. By tapping (or holding down) F9 you can make the numbers increase. Make sure students can understand the process: each ten ones increases the tens by 1 , and the ones drops back to 0 .
Skip count to billions
This runs like the odometer, but this time the size of each digit in the number is shown by the position of a coloured rectangle. Use F9 to change the number; each digit rises to 9 and then drops back to 0 as the place value goes into action. It is possible to start at any number, and to go up by any number. (It is fun to see patterns such as going up by 9 , or 11, or 999.) By typing a negative step size you can also make the numbers go down.
Decimal counting
By tapping F9 'Tenths' shows the value (in the diagram and the number) increasing by 0.1 at each tap.

- When you get to ten tenths, a new 1 appears.
- When you reach 9.9 a TEN appears, and so it continues.

By tapping F9 'Hundredths' shows the value (in the diagram and the number) increasing by 0.01 at each tap.

- When you get to ten hundredths, a new tenth appears.
- When you get to ten tenths, a new 1 appears.
- When you reach 9.9 a TEN appears, and so it continues.


## Tenths by tenths

This does what is described above as 'Multiply two decimals'.

## b Multiples and multiplication tables

## Rote learning of multiplication facts

Although remembering facts is not about understanding, students need to both understand multiplication and also remember all the facts so they can move forward. Card games are very useful as an enjoyable way to help recall.
1 Card games
very useful as an enjoyable way to help recall.

## Card game: Multiplication war

A small group (three or four) split the cards (1 to 10 only, no pictures) evenly. Stack them upside down.
Each person turns over two cards multiplies them and says the answer. This is checked by the others. The person with the biggest answer wins all the pairs of cards. If two or more have the same answer, then only those play again in a 'war', and the winner gets all the pairs of cards from all other players. At the end the winner has all the cards.

## Card game: Multo

You can play this game with the whole class, in a small group or even by yourself.
Everyone who is playing draws a
$4 \times 4$ square and writes 16 different product numbers to multiplications from $1 \times 1$ to $9 \times 9$.
Here is an example.

| 24 | 25 | 30 | 42 |
| :---: | :--- | :--- | :--- |
| 3 | 18 | 27 | 1 |
| 8 | 9 | 36 | 81 |
| 56 | 49 | 40 | 15 |

Use only the numbers 1 to 9 from a pack of cards. Someone shuffles the cards and you take two at a time to make the multiplication question, such as $3 \times 7$. (If you forget the answer, you may ask someone.)
If the number is one of your 16 products, you cross it out. Continue until one person wins with four in a line - a row, a column or a diagonal. At that point they yell "MULTO".

## 2 LCM - using bottle top rectangles

Each group needs about 30 bottle tops. Bottle tops are made into rectangles.
1: Find the first four numbers of bottle tops that can be put into rows of 2 and also into rows of 3 .
(The numbers are $6,12,18,24$. These are the multiples of 6 , so 6 is the LCM of 2 and 3 .)
2: Find the first four numbers of bottle tops that can be put into rows of 2 and also into rows of 6 .
(The numbers are $6,12,18,24$. These are the multiples of 6 , so 6 is the LCM of 2 and 6 .)
3: Find the first four numbers of bottle tops that can be put into rows of 4 and also into rows of 6 .
(The numbers are $12,24,36$. These are multiples of 12 , so 12 is the LCM of 4 and 6.)

## Card game: Find the lowest common multiple

You need a pack of playing cards. Shuffle the cards. Aces show 1, J = 11, Q = 12, K=13.
On your turn, you turn over two cards and work out the LCM of the two numbers (e.g. 4 and 7). You get a point if you can show why you are correct, and lose a point if the other players can show why you are wrong.

## LCM

You need a pack of cards. Keep only the $2 \mathrm{~s}, 3 \mathrm{~s}$ and 5 s . There are four of each.
Shuffle the cards. Put out three cards where you can see them. The cards are the prime factors of a number.
Write down the factors and work out the number. For example $2,3,5$ means $2 \times 3 \times 5=30$. Shuffle the cards and repeat for the second number.
Shuffle and repeat for the third number.
Find the LCM of the three numbers.
Then it is the second player's turn to do the same. Then the third player, and so on. When all have finished, the player with the smallest LCM wins.

## Spreadsheets

## Multiples

This arranges the numbers in 25 rows. If there are 10 columns it goes to 250 , but you can change the number of columns.
It colours the multiples of numbers you choose, red for one colour and blue for another. Numbers that are multiples of both are coloured purple. Look for the patterns of the colours, and try to explain why they occur.

## c Factors and division

## 1 Base 10 sheets for dividing

We divide by splitting a number into multiples of the divisor.
For example, $36 \div 3=30 \div 3+6 \div 3=10+2$. Here it is using base 10 strips and squares.


Here is another example. $42 \div 3=14$

 سهسصس صسسسه
$\qquad$
regroup one of the tens to get

## Card game: Division facts

You need a pack of cards, without 10s, Js, Qs or Ks. Shuffle the cards. Turn them upside down.
On your turn, put out three cards.
The first two make a two-digit number, and the third is the divisor.
To win the point you must correctly give the quotient (the division answer) and the remainder, if there is one.

## Card game: Divides

Shuffle the cards and invert the pile.

- On your turn you take three cards (or roll the three dice).
- You have to make a two-digit number from any two cards (dice) and divide by the third.
- You get one point for each exact division you can make, and show to the other players.

For example, 2,3 and 7 can make $27 \div 3$ ( $=9$ ) and $72 \div 3$ (= 24 ), for two points.

- Put the used cards into another pile.

When you run out of the first pile, shuffle again and continue. The winner has the highest score.
2 Factors
If you have a number of things you can sometimes put them into rows and columns to make rectangles. You could do this as a demonstration or ask groups to sort numbers of bottle tops. When you have a rectangle, the number of bottle tops is a multiple, and the length and width of the rectangle are factors. These show $12=3 \times 4,6 \times 2$ and $1 \times 12$.


Factors are also the whole numbers that divide exactly into another whole number. They are sometimes called divisors.

## Card game: Factors

You need two to five players. Use numbers 1 to 9 from a pack of cards and let 10 mean 0 , or a set of digits 0 to 9 . Copy the score sheet below.
In this game you win if your two-digit number has the most factors. Shuffle the cards. Each player chooses two cards and puts them into the spaces opposite their name (see below). Name Number Factors

## Nasty factor game

This game is like the one above, but you take turns to choose one number and put it into either your number or another player's number.
This means you can be nasty! You make a number with a small number of factors for others, and win.
3 Common factors
Use set of bottle tops. For example the common factors of 8 and 12 are 2 and 4, because they both can be made with 2 rows ( $2 \times 4$ and $2 \times 6$ ) and 4 rows ( $4 \times 2$ and $4 \times 3$ ).
4 Highest common factor (HCF)
Use sets of bottle tops. For example you can find the HCF of 8 and 12 by putting 8 tops into 4 rows, and 12 into 4 rows. It cannot be done with a higher number, so 4 is the highest factor common of both numbers.
$8=4 \times 2$


## Card game: HCF

You need a pack of cards. Keep only the $2 s, 3 s$ and 5 s. There are four of each.
Shuffle the cards. Put out five cards where you can see them. The cards are the prime factors of a number.
Write down the factors and work out the number. For example 2, 3, 5 means
$2 \times 3 \times 5=30$.
Shuffle the cards and repeat for the second number.
Shuffle and repeat for the third number.
Find the HCF of the three numbers.
Then it is the second player's turn to do the same. Then the third player, and so on.
When all have finished, the player with the biggest HCF wins.

## Card game: Remainder game

Two or more players use the same board, moving their own counter. Everyone starts bottom left, on 13. You must get to, or pass, WIN to win.
Remove the picture cards from a pack. On your turn you shuffle the pack of cards and choose one. divide the number you are on by the number you choose, e.g. $13 \div 6$. You then move forward (following the arrows) by the remainder. (For $13 \div 6$, the remainder is 1 so you move to 17.) Sometimes you cannot move, because the number you choose is a factor of the number you are on.


## Spreadsheets

Common factor
This shows two numbers that have a common factor. The student types the common factor.
The graph may also help. (The common factor is the value of y on the graph where $\mathrm{x}=1$.)
Remainder game
This is a digital version of the game described above. It may be played by one or more students. The student needs to get both parts of the division answer correct to move the 'counter' (the yellow-coloured rectangle).

## d Order of operations

Students must understand that BODMAS, or variations, can be misleading. If you always perform additions before subtractions you can be wrong. The better understanding is this:

- Perform operations in brackets
- then any powers (indices)
- then all multiplications or divisions in the order they come from left to right
- and lastly all additions or subtractions in the order they come from left to right.


## Card game: Got it

Many players can play this at once. Use cards for numbers 1 to 10 . This is a very good game for developing creativity with number facts, and using the order of operations.
Deal six cards to each player, invert the rest, and turn over the top card. This will be the target number.
At the same time, each player tries to use as many of their cards as possible to make an expression that is equal to the target number. You show the others, get your points, and then you can use the cards again.
Two-digit numbers may be used using two cards.
Players get one point for each card used in their expressions. When each player has done their best, put all the cards together, shuffle and deal out again.

## Card games: Make 1 to 10

You need a pack of cards. Use only 1 to 9 .
Game 1:
Each person writes these numbers on small pieces of paper: 1, 2, 3, 6 .
Shuffle the cards. Turn over the top card. This is the answer. Use all of the numbers 1, 2, 3 and 6 in any order and once only, with or without brackets and the four major operations as needed, to make the number that appears. The winner is the first to make the expression, but you must write it correctly and explain it to the other players to earn the point. When all players have earned their points, turn over another answer and try again.
Game 2: Use 1, 3, 5 and 7
Game 3: Use 1, 3 and 5.
For this some numbers cannot be made. So, if 4 turns up make 12, if 5 turns up make 14 and if 6 turns up make 16.
Game 4: Use 1, 2, 3 and 4.
For this some numbers cannot be made. So, if 7 turns up make 10, if 8 turns up make 13 and if 9 turns up make 20.
Game 5: Use 4, 4, 4 and 4.
Make as many target numbers as you can using all four 4 s .
Game 6: Use 3, 3, 3 and 3 (four 3s).
Make as many target numbers as you can using all four 3 s .
Game 7: Use 2, 2, 2 and 2 (four 2s).
Make as many target numbers as you can using all four 2s.

## e Fractions

## Game - Estimating fractions on a string

You need about 2 or 3 m of string, ribbon or rope and some clothes pegs. Two people hold the string tight. Choose one end to be 0 and the other end to be 1 .
Choose a fraction, such as $\frac{3}{4}$. Then each person in the group puts a peg where they think that fraction comes on the string.
To find who is closest, fold the string into the correct number of equal parts.
Then choose a different fraction and try again.

## 1 Fraction strips

We are going to work with strips of paper that are all the same length. The easiest is the length of an A4 sheet. This will be the 1.


There will be two ways we can divide this length into equal parts: folding or marking. Folding
You can make strips with each of the fractions marked by folding it lengthways (as in the picture in the book)


It is best for students to fold these themselves and to draw each fold line after they have opened out the strip. They should write the size of each fraction on their strips.
These will be used for adding and subtracting, but first they must understand equivalent fractions.

## Card game: Fraction war

Up to 5 players. Shuffle the cards (1 to 10), and share the card pack equally.

- Turn cards upside down. Each player chooses two cards to make a fraction with smaller over bigger. Two equal numbers are permitted.
- The biggest fraction in the group wins. The winner gets the cards from all the others. Two equal numbers makes the fraction 1.
- If two or more are the same size, those people have a war! Only they choose two more cards. The bigger fraction gets all the cards from everyone
- At the end, the person with the most cards wins.


## 2 Equivalent fractions

Use the fraction strips (see template at end of this document). Any two fractions that are the same length are equivalent.
Students will soon see that equivalent fractions are obtained from the simplest form by multiplying numerator and denominator by the same number. Why is this true? It is because when you, for example, double the denominator there are twice as many parts, so you need twice as many to make the same length of strip - which means you must double the numerator.

| $\frac{1}{8}$ |  | $\frac{1}{8}$ |  | $\frac{1}{8}$ |  |  | $\frac{1}{8}$ | $\frac{1}{8}$ |  |  | $\frac{1}{8}$ | \% |  | $\frac{1}{8}$ |  |  | $\frac{1}{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{4}$ |  |  |  |  | $\frac{1}{4}$ | $\frac{1}{4}$ |  |  | $\frac{1}{4}$ | , |  |  |  |  |  | $\frac{1}{4}$ |  |
| $\frac{1}{2}$ |  |  |  |  |  |  |  |  |  |  |  | $\frac{1}{2}$ | $\frac{1}{2}$ |  |  |  |  |
| 1 whole |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| ${ }^{\frac{1}{3}}$ |  |  |  |  | ${ }^{\frac{1}{3}}$ |  |  |  |  |  | $\frac{1}{3}$ |  |  |  |  |  |  |
| $\frac{1}{6}$ |  | $\frac{1}{6}$ |  |  | $\frac{1}{6}$ |  |  | $\frac{1}{6}$ |  |  | $\frac{1}{6}$ |  |  |  | $\frac{1}{6}$ |  |  |
| $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |  | $\frac{1}{12}$ | 12 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |  |  | $\frac{1}{12}$ |  | $\frac{1}{12}$ |  | $\frac{1}{12}$ | $\frac{1}{12}$ |
| $\frac{1}{9}$ |  |  |  | $\frac{1}{9}$ |  | $\frac{1}{9}$ |  | , | $\frac{1}{9}$ |  |  | $\frac{1}{5}$ |  |  | $\frac{1}{5}$ |  | $\frac{1}{9}$ |
|  | , |  |  | $\frac{1}{3}$ |  |  |  |  |  |  |  | $\frac{1}{3}$ |  |  |  |  |  |
| $\frac{1}{10}$ | $\frac{1}{10}$ |  | $\frac{1}{10}$ |  | $\frac{1}{10}$ |  | $\frac{1}{10}$ | $\frac{1}{10}$ |  | $\frac{1}{11}$ |  |  | $\frac{1}{10}$ |  | $\frac{1}{10}$ | , | $\frac{1}{10}$ |

3 Add or subtract
Example: $\frac{1}{2}+\frac{1}{3}$.
We recommend demonstrating this first. It would be good to have a very large strip that can be easily seen from the back of the class. If this is not possible have the students stand around a table to see what you are doing with the strips.
You simply fold the half strip so you can only see the half. At one end you put the thirds strip, folded into three parts, so you can only see the third.

| $\frac{1}{2}$ | $\frac{1}{3}$ |
| :--- | :--- |

The problem is what to call the answer. For this we need a common name for each part.
Sixths is obvious, because $\frac{1}{2}=\frac{3}{6}$ and $\frac{1}{3}=\frac{2}{6}$.
So we put the sixth strip next to the others to get this. The answer is clearly $\frac{5}{6}$.


We can set it out this way: $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$
Please note: The way that fraction adding and subtraction is being set out here is NOT the traditional way in Malawi. Instead, we are using a way of setting out the written work that emphasises the finding of equivalent fractions.
Example: $\frac{1}{2}-\frac{1}{3}$.
Here is $\frac{1}{2}$ again, and we have put the $\frac{1}{3}$ on top of part of it.


Putting the sixths under these shows that the missing bit - shown by the arrow - is $\frac{1}{6}$.
We can set it out this way: $\frac{1}{2}-\frac{1}{3}=\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$.

## 4 Multiply with a rectangle

We can show a quick way to multiply, by getting a fraction of a fraction. We split a rectangle in two ways; it is very similar to multiplying decimals. Here is an example.
$\frac{3}{4} \times \frac{2}{3}$ means $\frac{3}{4}$ of $\frac{2}{3}$. Firstly split the rectangle into three parts vertically and shade the $\frac{2}{3}$.


Then get $\frac{3}{4}$ of $\frac{2}{3}$. Split the rectangle into 4 parts horizontally and shade the $\frac{3}{4}$.


Each little rectangle is one twelfth (the same as $3 \times 4$ ) and the $\frac{3}{4}$ of $\frac{2}{3}$ (shaded both ways) is 6 of them. So $\frac{3}{4}$ of $\frac{2}{3}=\frac{6}{12}=\frac{1}{2}$ (in simplest form).

## Spreadsheets

## Equal fractions (lines)

A fraction is given and the denominator of a second fraction is given. On the graph the length of 1 displays a line the length of the fractions, and divides the one into the lengths of the second fraction. This helps the student determine the numerator (the number of parts that are the same length as the first fraction).

## Add strip fractions

'Add' shows the two fractions end-to-end. They are already converted to equivalent fractions (with the same denominator) so you just have to add the numerators. the diagram shows what is going on.

## Subtract strip fractions

'Subtract' shows the first fraction above the second. the answer is the difference.

## Fractions of a rectangle

This draws a rectangle already divided as in 'multiply with a rectangle' above. It shades the second fraction and the doubly shades (in yellow) the first fraction of the second fraction. So the answer is the yellow part as a fraction of the whole rectangle.

## f Percentages

## 1 Relate percentages to fraction strips

The picture of the fractions strips below has a black decimal and percentage line between the white sets of fractions (see template at end of this document). The aim of having student use this tool is to show them that percentages are just another type of fraction, as are decimals, and help them see the links.


## 2 Grid paper

The $10 \times 10$ square (see template at end of this document) has 100 little squares, and so it shows $100 \%$. It is also 1.
So $\frac{3}{4}$ of the square is $75 \%$ and so on. Note that it is always possible to have more than $100 \%$; many students believe you cannot go over $100 \%$, basing their ideas on test results.


6\% (0.06) and 32\% (0.32)


288\% (2.88)

## Game - Estimating percentages

You need a pack of cards. J = 11, Q = 12 and $\mathrm{K}=13$. Shuffle the cards.
Choose any two cards, and make a fraction from them (in either order).
All players estimate the fraction as a percentage. Work it out with a calculator, and the person nearest the correct answer wins a point.

## 3 Metric tape

- If you have a metric tape going up to about 5 metres, you can relate percentages to the number of centimetres. Each metre has $100 \%$, so 1.25 m is $125 \%$ of 1 metre.
- Draw two lines of any length on paper.

Call the first one $100 \%$. Estimate the length of the other.
Then call the second one $100 \%$ and estimate the length of the first.

- Choose one special length in your classroom to be $100 \%$. Choose other different lengths around your classroom and estimate them as percentages of the special length. Measure and divide to check.


## 4 Percentage increase or decrease

You will find that diagrams are the most useful ways to help students to visualise this difficult topic. For example a price has changed from K290 to K380. What is the percentage increase? The change is always a percentage of the ORIGINAL value

The increase is K380 - K290 = K90.


K90 as a percentage of K290 $=90 \div 290=$ $9 \div 29=0.31=31 \%$.

## Card game: Estimating percentage increases or decreases

You need a pack of cards. Use $\mathrm{J}=11, \mathrm{Q}=12$ and $\mathrm{K}=13$. Shuffle the cards.
Choose two cards and make a two-digit number. This is your old price, in kwachas. Choose another two cards for another two-digit number - this is your new price. The difference is increase or decrease. All players estimate this as a percentage of the old price. Work it out with a calculator, and the person nearest the correct answer wins a point.

## Spreadsheets

Percentage full
A glass is pictured, partly full of red liquid. You must estimate the percentage. You may be up to 5\% out. the correct answer is shown. Delete your guess for another problem.
Percentages of money
You are presented with a problem. You only need to estimate the answer. The graph may help you picture the required percentage of the amount shown.

## Mark up

The original price and the increased price are given and graphed, the mark-up (increase) is the amount of increase. You type the amount of mark up and the mark up as a percentage of the original price.
Discount
The original price and the decreased (discounted) price are given and graphed, the discount (decrease) is the amount of decrease. You type the amount of discount and the discount as a percentage of the original price.

## 3 Hands-on methods in Form 1

## Introduction

Many Form 1 students have little real understanding of maths. Many of them probably have no idea that maths can be understood, as this possibility has not been previously presented to them. The purpose of these activities is to open a whole new world of understanding and enjoyment to them. We are working to change the attitude of students to the learning of maths, so they enjoy the experience and become highly engaged.
Matching to the syllabus
The present Form 1 (and 2 ) syllabus presents topics in a disconnected way. This book shows the natural links between topics, to make the ideas easier to understand.

- Number systems: Bases page 16, Integers page 21
- Algebraic expressions: page 24. This is part of Algebra, not Number
- Simple algebraic fractions: page 32
- Approximations and estimations: page 19
- HCF and LCM: page 6-10
- Factorisation: page 32
- Commercial arithmetic: page 20
- Linear equation: page 28
- Number patterns: page 24. This is a good place to start for Algebra.
- Sets: pages 17 and 36. I have used this to revise multiples, factors and shapes.
- Transformation: page 39. Clearly part of Geometry.
- Lines and angles: page 33.
- Triangles and polygons: page 33-39
- Geometric constructions: page 33
- Co-ordinate geometry: page 26
- Statistics: page 40


## A Number

## a Bases other than ten

## 1 Bundles to count in base 3

When we count in ordinary numbers the magic (base) number is ten. We group ones into tens. If there are enough tens we group them into hundreds.
In base 3 , the magic (base) number is three. We group ones into three. If there are enough threes we group them into nines. This number is 2 (nines) 1 (three) and 2 (ones).


Let groups count numbers of bottle tops, maize seeds, or other objects, by grouping as above.

- To add in base 3 , just make the numbers, combine them and then count the answer.
- To subtract in base 3 , just make the larger number, take away the number of objects for the smaller number, and count the answer.


## Spreadsheets

Ten-frame counting
This uses ten-frames ( $2 \times 5$ rectangles) to count ones, grouping them into tens, and then grouping the tens into hundreds. the same idea occurs in other baes.

Base 5
This sets the student keep adding 1 to see five 1 s become 10 (in base 5), and eventually five 10 s become 100, and five 100s become 1000. It goes up to 4444, then starts again.

## b Sets (of multiples and factors)

1 Use sets of multiples
Give each group a set of numbers on slips of paper: $1,2,3,4,5,6,7,8,9,10,11,12$.
Ask them to demonstrate these facts using the numbers.

- Multiples of 6 are a subset of multiples of 2 .
- Multiples of 2 and multiples of 3 have multiples of 6 as the intersection. ( 6 is the LCM.)
Ask them to find other examples.


## 2 Use sets of factors

Give each group a set of numbers on slips of paper: $1,2,3,4,5,6,10,12,15,20,30,60$. Ask them to demonstrate these facts using the numbers.

- Factors of 6 are a subset of factors of 60.
- Factors of 12 and factors of 15 have factors of 3 as the intersection. ( 3 is the HCF.) Ask them to find other examples.


## Spreadsheets

## Venn and grid diagrams - multiples

Using only numbers 1 to 20 , this sorts numbers by multiples of each of two numbers. The intersection is shown. Students answer many questions. UsE F9 to change the two numbers. Venn and grid diagrams - factors
This sorts the factors of each of two numbers. The intersection is shown. Students answer many questions. Using F9 changes the two numbers used.

## c Decimal division

Place value, adding subtracting and multiplying are in the primary revision section. This deals with division by a decimal number. The basic idea is the multiplying both numbers in a division by 10,100 , etc. will not change the answer. we keep doing this until the divisor is a simple whole number.
For example $0.12 \div 0.006=1.2 \div 0.06=12 \div 0.6=120 \div 6$. So the answer is 20 .
1 Base 10 sheets for dividing by a decimal
We use the base 10 place value sheets.


One

one tenth one hundredth

When dividing by a decimal, we must use the meaning of division that asks how many times the divisor fits into the first number. So for $0.12 \div 0.04$, we ask how many 0.04 s are in 0.12 .


There are 3 . So $0.12 \div 0.04$ has the same answer as $12 \div 4$.
Students need to explore this idea using a variety of questions and the hands-on material.
Possible questions to use:

| 1 a $1.02 \div 6$ | b $1.14 \div 6$ | c $1.38 \div 6$ | d $1.44 \div 6$ | e $1.5 \div 6$ |
| :--- | :--- | :--- | :--- | :--- |
| 2 a $0.91 \div 7$ | b $1.05 \div 7$ | c $1.26 \div 7$ | d $1.04 \div 8$ | e $1.12 \div 8$ |
| $31.2 \div 0.3$ | b $1.2 \div 0.6$ | c $1.2 \div 0.4$ | d $1.2 \div 0.2$ | e $1.2 \div 0.1$ |
| 4 a $1.4 \div 0.4$ | b $1.4 \div 0.5$ | c $0.6 \div 0.5$ | d $0.6 \div 0.8$ | e $0.63 \div 0.7$ |
| 5 a $0.2 \div 0.5$ | b $0.3 \div 0.5$ | c $0.4 \div 0.5$ | d $0.6 \div 0.5$ | e $0.7 \div 0.5$ |
| 6 a $0.2 \div 0.8$ | b $0.4 \div 0.8$ | c $0.6 \div 0.8$ | d $1 \div 0.8$ | e $1.4 \div 0.8$ |

## Games - Dividing by decimals

You need a pack of cards. There are several games depending on which cards you leave out. The Aces show 1.

## Game 1

Take out 7, 8, 9, 10, J, Q and K. So you use only the numbers 1 to 6 .
The players choose one of the expressions A to L to use for the game.
A $60 \div 0 . \square$
B $6 \div 0 . \square$
C $0.6 \div 0 . \square$
D $0.06 \div 0 . \square$
E $60 \div 0.0 \square$
I $60 \div 0.00 \square$
F $6 \div 0.0 \square$
J $6 \div 0.00 \square$
G $0.6 \div 0.0 \square$
H $\quad 0.06 \div 0.0 \square$

Shuffle the cards. On your turn you turn over one and put it in place of the square. Work out the answer, and explain it to the others.

## Game 2

Leave out 7, 9, 10, J, Q and K. So you use the numbers 1 to 6 and 8 . Use one of the expressions $A$ to $L$. Use the same method to play.

| A $120 \div 0 . \square$ |  | $\div 0 . \square$ |  | $1.2 \div 0$. |  | $0.12 \div 0$. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| E 120 $\div 0.0 \square$ | F | $2 \div 0.0 \square$ | G | $1.2 \div 0.0$ | H | $0.12 \div 0.0$ |
| $120 \div 0.00 \square$ | J | $2 \div 0.00$ | K | $1.2 \div 0.0$ |  | $0.12 \div 0.00$ |

## Game 3

Leave out 9, 10, J, Q and K. So you use the numbers 1 to 8 . Use one of the expressions A to J. Use the same method to play.
A $840 \div 0 . \square$
E $840 \div 0.0 \square$
I $840 \div 0.00 \square$
B $84 \div 0 . \square$
C $8.4 \div 0$.
$\square$
D $\quad 0.84 \div 0 . \square$
J $84 \div 0.00 \square$

## Spreadsheets

Divide by tenths
There are three tabs.

- 'Up to $9 \div$ tenths' draws the graph, and also converts the numbers to make a whole number divisor.
- 'Up to $99 \div$ tenths' does the same with bigger numbers.
- 'Less than $1 \div$ tenths' will always have answers less than 1 , and the answer will be a number of tenths.


## d Rounding, estimation and significant figures Rounding

The point of rounding is that some answers must be quoted to fewer decimal places than is strictly accurate, for practical purposes of measuring. Indeed, some answers can never be expressed exactly using the decimal system (e.g. repeating decimals, value of $\pi$ ) and these ideas need to be discussed as part of the teaching.
1 Measuring out areas and lengths
These are outdoor activities. Each group needs their own area in which to work.
a Rectangle
The group locates a base line 3 m long, the width of the rectangle. They then measure out the length so that the area will be 10 square metres. The length required is $3 \frac{1}{3}$ metres, but measuring this requires rounding to 2 or 3 decimal places ( 3.33 m or 3.333 m ).

## b Square

The group find the corners of a square with area will be 10 square metres. The length required is $3.16227766 \ldots$ metres, but measuring this requires rounding to 2 or 3 decimal places ( 3.16 m or 3.162 m ).
c Distance rolled by a bicycle wheel in one turn
As it rolls, a bicycle wheel rolls a length equal to its circumference. Students measure the diameter of a wheel (about 66 cm ), and predict how far it will roll ( $66 \times \pi=2076.3 \mathrm{~cm}$ ). They measure this out and then roll the wheel to check.

## Estimation for multiplication

The other reason for rounding is to permit us to mentally predict the results if a calculation. Rounding to one non-zero digit (one significant figure) is the most useful for estimation of multiplications.

## Card games: Estimating decimal multiplication

Here are four games in one. You need a pack of playing cards. The Ace is 1 and the $10, \mathrm{~J}, \mathrm{Q}$, K are all 0 . On a sheet of paper, write one of these multiplications, with spaces big enough to put playing cards.
А $0 . \square \square \times \square \square$ B
B $0.0 \square \square \times \square 0$
c $0 . \square \square \times \square . \square$
D $0.00 \square \square \times \square \square 00$ Choose one of them for each game. Someone puts out four cards into the spaces and each player estimates the answer as closely as possible. Then you all work out the answer, and the closest wins.

## Estimation for division

To estimate divisions, I suggest this method: example $4.72 \div 0.735$
Round the divisor to one significant digit: example $4.72 \div 0.7$
Round the first number to the nearest multiple of the divisor: example $4.9 \div 0.7$
If needed multiply or divide both numbers by 10 enough times to make the divisor a single digit whole number, and calculate: example $49 \div 7=7$

## Card games: Estimating decimal division

Here are four games in one. You need a pack of playing cards. The Ace is 1 and the $10, \mathrm{~J}, \mathrm{Q}$, K are all 0 . On a sheet of paper, write one of these multiplications, with spaces big enough to put playing cards.
A $0 . \square \square \div \square \square$ в
B $0.0 \square \square \div \square 0$
C $0 . \square \square \div \square . \square \quad \mathrm{D}$
$0.00 \square \square \div \square 00$ Choose one of them for each game. Someone puts out four cards into the spaces and each player estimates the answer as closely as possible. Then you all work out the answer, and the closest wins.

## Spreadsheets

## Rounding whole numbers

This gives a whole number and shows the position of that number on several number lines. It asks the user to type the number rounded in some way.
Rounding decimals
This gives a decimal number and shows the position of that number on several number lines. It asks the user to type the number rounded in some way.

## Estimate to multiply

This shows a simple multiplication, and gives the single digit rounded version; you type the estimation.
Division fact rounding
For a given division question, this shows the process of estimation described above.

## e Social and commercial arithmetic

The main purpose of this section of the syllabus is to help students to learn to manage their money transactions. At Form 1 none of this is personally relevant, unless the student is already an adult. But some of it will be very relevant to their extended family. many parents will not have had the opportunity to learn about these things, so take the step of asking students to explain them to their parents.
The topic of the syllabus are these: Tax (VAT, PAYE), Insurance, Bills \& budgets, Inflation, Devaluation, Currency exchange, Depreciation, Investments.
Your challenge is to explain these complex financial processes to students who have little background in the meanings of the words you will use. It is suggested that you try to use role-play (pretend) transactions to help them visualise what is going on.
1 Diagrams
Start with one they know well - buying goods at a shop. Introduce the proper English words for the people involved (shopkeeper, customer) and the elements of the purchase (cost price to the shop, VAT, profit). It might be useful to show the amount paid broken up into its components (cost, VAT, profit) and to discuss the expenses that a shopkeeper has to pay such as salaries, rent, money kept to buy new stock. This can be done with a simple rectangle. They might estimate the percentage of the payment that goes to each component.
PAYE tax means that the stated amount of pay is not what a worker actually gets, as some is kept back to pay the government tax department. Discuss the percentage of the amount paid that must always go to tax.
The other topics need to be presented as dramatically as possible, so that they are both understood and remembered into later life.

## Spreadsheets

## Budget

This presents the two sides (income and expenses) and converts amounts that are not 'per month' into monthly figures. The pie graph shows how the expenses are split. the user simply enters the numbers of kwachas into the cells and checks whether or not the planned use of the money is adequately covered by the income.
Profit or loss
This randomly chooses buying prices, expenses and selling prices. Students calculate the profit or loss, and estimate it as a percentage of the buying price.

## Commission

This describes retainer and commission as part of the total wage. The commission is a percentage of sales, and this is all shown as bar graphs.
Mark-up
The buying price and the (higher) selling price are shown on the graph. You enter the percentage mark-up.
Simple interest
The amount in the bank (principal) and the interest rate determine the amount of simple interest added to your account for a number of years. This is shown on the graph.

## Discount

Discount is a kind of negative mark-up. The marked price is reduced to the sale price, using a percentage. This is shown on the graph.

## f Directed number (integers)

## 1 Walking left or right to add integers

This relates the idea of integers to changes in position on a number line. It is good for each group of students to have their own space and their own number line.

The number line is labelled by the position you get after you take a number of steps forward from zero while facing right - for positive, and left - for negative.

| -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |

To add two integers you just start at zero and do one walk after another.
For example here is $-3+4$.
The first number is negative; face left and walk 3.
The second is positive; face right and walk 4 . We end up at 1, which is the answer.


Make up a series of additions for groups of students to work out together using walks.

## 2 Using bottle tops to add integers

Explanation for teacher:
They can be smooth side up for positive ( $\boldsymbol{O}$ ) or rough side up for negative ( O ).
Adding positive is just like taking steps to the right, and adding negative is just like taking steps to the left. So opposites add to zero.
This means that there are many ways to show 0 , such as these.


```
-00000
```

Here are some ways to show ${ }^{-2}$.


Here are some ways to show ${ }^{+3}$.

```
00000000 0000000
```


## Bottle tops to add integers

To add we just make the two integers, and combine them. Then opposites add to zero.
For example, -3+2 = -1.
Make up a series of additions for groups of students to work out together using bottle tops.

## Card game: Add integers

Use only the cards 1 to 10 , and take 10 to mean 0.
Black cards are positive numbers (or zero). Red cards are negative numbers (or zero).
Split the cards evenly among up to six players. They have the cards upside down in front of them. They take turns to turn a card over and ADD its number to the total already made by the other players. If the total returns to zero, the person who makes it happen wins a point. The game continues until players run out of cards. Most points wins.

## 3 Using walks to subtract integers

To subtract two integers you just start at zero and do one walk after another, BUT YOU WALK BACKWARDS FOR THE INGTEGER BEING SUBTRACTED.
For example here is $-3-(-4)$. The first number is negative; face left and walk 3.
The second is negative so face left, but it is being subtracted, so WALK BACKWARDS; face left and walk backwards 4.
We end up at 1 , which is the answer. We see that subtracting a negative number has the same result as adding its opposite.

| -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 |  | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |

Make up a series of subtractions for groups of students to work out together using walks.
4 Bottle tops to subtract integers
Explanation for teacher:
You will remember that we can add pairs of opposites to make different versions of the same number. This is useful!
Let us look at $-2-+2$.
Here are some ways to show ${ }^{-2}$.
000000000000
The one on the right has +2 so we can subtract 2 from it. Removing - leaves only OOOO, which is -4 . So $-2-+2=-4$.
This is like adding -2 , since $-2+-2=-4$.
Let us look at ${ }^{+} 2-{ }^{-2}$.
Here are some ways to show ${ }^{+2}$.

The one on the right has ${ }^{\text {c }} 2$ so we can subtract -2 from it. Removing OO leaves only - - - , which is ${ }^{+} 4$. So ${ }^{+2}-{ }^{-2}={ }^{+} 4$.

This is like adding ${ }^{+} 2$, since ${ }^{+} 2+{ }^{+} 2=+4$.
In both cases, subtracting the integer is like adding its opposite.
Make up a series of subtractions for groups of students to work out together using bottle tops.

## Card game: Subtract integers

Use only the cards 1 to 10 , and take 10 to mean 0.
Black cards are positive numbers (or zero). Red cards are negative numbers (or zero).

Split the cards evenly among up to six players. They have the cards upside down in front of them. They take turns to turn a card over and SUBTRACT its number from the total already made by the other players. If the total returns to zero, the person who makes it happen wins a point. The game continues until players run out of cards. Most points wins.

## 5 Using walks to multiply integers

Multiplying by a positive number means repeated addition.
So $2 \times 3=6$, and $2 \times-3=-6$.

Multiplying by a negative number means repeated subtraction.
So $-2 \times 3=-6$ (Start at zero, face right ( +3 ) and walk backwards three steps, twice).
$\begin{array}{lllllllllllll}\leftarrow & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6\end{array}$

Make up a series of multiplications for groups of students to work out together using walks.

## 6 Bottle tops to multiply by integers

Explanation for teacher:
Multiplying by a positive number is the same as repeated addition. For example $3 x-2=-2+-2+-2=-6$. It is shown this way:


Multiplying by a negative number is the same as repeated subtraction from zero.
To subtract three lots of -2 from 0 , we need this way to show 0 .

- $0 \cdot 0 \cdot 0$ QOOO

So $-3 x-2=0--2--2--2=6$.
Make up a series of multiplications for groups of students to work out together using bottle tops.

## Card game: Multiply integers

Use only the cards 1 to 10 , and take 10 to mean 0.
Black cards are positive numbers (or zero). Red cards are negative numbers (or zero).
Split the cards evenly among up to six players. They have the cards upside down in front of them. They take turns to turn two cards over and say its product (positive or negative). The greatest product wins all the cards from the others. Continue in this way until one player has all the cards.

## Spreadsheets

Walk the plank
This shows a person walking on the number line - the 'plank'. Tap F9 to see what they will do, say aloud what you think will happen, and then tap F9 again to see what does happen. The moves will add or subtract positive or negative numbers.

## Integers

This shows adding (red and green squares with opposites cancelling out.
It shows subtracting by red (negative) for subtracting a positive, and green (positive) for subtracting a negative.
For multiplication it shows positive (green) for multiplying two positives or two negatives, but red (negative for multiplying two different signs.

## B Algebra

## Introduction

Algebra is often taught like a foreign language, but with little attempt at making sense. the approach followed here shows that algebraic expressions are generalisations of number patterns. So whatever happens with numbers also happens when you replace the numbers with letters.

## a Number patterns and algebraic expressions

It is best to start with number patterns that learners can understand, but introduce the idea of a generalised expression to describe the pattern. Some people like to call it the formula (or rule) used to make the numbers.

## 1 Making your own number patterns

## Card game: Make a number pattern

Use cards to select a number to add (if black) or subtract (if red).
Use cards to select a number to multiply by positive (if black) or negative (if red).
Decide to either of these:

- multiply first, then add or subtract, or
- add or subtract first then multiply

To make the pattern use the sequence $0,1,2,3$ in a table. Describe the pattern. Here is an example:

| sequence | 0 | 1 | 2 | 3 | algebra |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\times 2$, then add 5 | $0 \times 2+5=5$ | $1 \times 2+5=7$ | $2 \times 2+5=9$ | $3 \times 2+5=11$ | $n \times 2+5$ |
| add 5 then $\times 2$ | $(0+5) \times 2=10$ | $(0+5) \times 2=12$ | $(0+5) \times 2=14$ | $(0+5) \times 2=16$ | $(n+5) \times 2$ |

Once learners have seen the pattern in the number expressions they can use letters such as $n$ (for number) to 'generalise' those expressions: $n \times 2+5$ for the first, and ( $n+5$ ) $\times 2$ for the second. The 'usual way' to write these expressions is $2 n+5$ and $2(n+5)$.

## 2 Match stick patterns

Each group needs about 20 match sticks. They should try to work out the number expression for each pattern and generalise it using algebra.
For example \#3 (row of squares) is $w \times 3+1=3 w+1$. They will see that the multiplier (3) appears in the number patterns as the 'step size', and the 1 is the value you need to add to get the correct answer after doing the multiplication.

A Create the shapes adding as you go, and from them, find the pattern of numbers.
B Describe the pattern and find the rule to get the number of pop sticks $(p)$ from the width of the pattern ( $w$ ).
C Explain the rule using the number pattern and the shape pattern.

$$
\text { Width } 1 \text { Width } 2 \quad \text { Width } 3
$$


$\begin{array}{lllll}\text { width (w) } & 1 & 2 & 3 & 10\end{array}$ number of matches ( $m$ )

Rule: $\boldsymbol{m}=$



3


$$
\begin{array}{lllll}
\text { width }(w) & 1 & 2 & 3 & 10 \\
\text { \# matches }(\boldsymbol{m}) \\
\text { Rule: } \boldsymbol{m}=
\end{array}
$$

4


> | width (w) | 1 | 2 | 3 | 10 |
| :--- | :--- | :--- | :--- | :--- | \# matches ( $m$ )

Rule: $\boldsymbol{m}=$

5


6


Rule: $\boldsymbol{m}=$

$$
\begin{aligned}
& \begin{array}{lllll}
\text { width }(w) & 1 & 2 & 3 & 10 \\
\text { \# matches }(m) \\
\text { Rule: } \boldsymbol{m}=
\end{array}
\end{aligned}
$$

## 3 Crossing the river

It is good to act out this little puzzle with four learners at the front of the class. Choose two larger learners to be 'adults' and two smaller ones to be 'children'. Explain the situation.

There are two adults and two children in a family who come to the bank of a river.
They wonder how they will get across. Then one of the children finds a small canoe.
However it is so small that it cannot hold two adults at once or even one adult and one child. It can hold one adult alone, one child alone or two children.
How many trips do they need to make?
Set this as a puzzle to be solved. They might need to use pencils and other objects to be the pretend adults and children.
Once they have the idea, ask the class to complete the table. (Below are the answers.)

| Number of adults with 2 children | 0 | 1 | 2 | 3 | 4 | algebra |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of trips across river | 1 | 5 | 9 | 13 | 17 | $4 n+1$ |

The 4 in the expression shows that you need 4 trips to get each additional adult across. The 1 in the expression is the first trip needed by the two children.
The graph may be drawn as soon as student know about coordinates. (See below.)

## Spreadsheets

## Match shapes

This draw the first three shapes of a match-stick pattern, with widths 1, 2 and 3 . You count the number of matches in each pattern, and type the number in the yellow cells in the table. By noting the step size (rate of increase) you find the multiplier in the expression, and use that to work out what is added. Type these numbers into the formula $m=$.

## b Graphs and gradients

## 1 Coordinates by playing Battleships

Each player needs a sheet of cm grid paper (see template at the end of this document). Each sets out two grids, with 0 to 9 on the horizontal axis and 0 to 90 on the vertical axis.



On one grid each player draws a fleet of ships using dots in a row - horizontally, vertically or diagonally: 1 battleship ( 5 dots in a row), 1 cruiser ( 3 dots in a row), and 2 destroyers ( 2 dots in a row) (see example above). They do not let their opponent see the positions.
Students compete in pairs, and they must be honest.
Take turns to fire torpedoes into the opponent's fleet, by naming a point.
If the torpedo hits a point where there is a ship, you must say 'hit' or 'miss'. Then the person firing marks that point (H or M) on their own empty grid, and continue until they can say the position of the opponent's fleet. Each ship sinks only when all its dots are hit.
Clearly the winner is the one who sinks all the 'enemy's' fleet first.

It is now a good time to graph the points from the Crossing the river puzzle. Here it is.
$t=4 n+1$


## 2 Stretching a rubber band

A spring or rubber band will stretch when a mass is hung on it. If the first masses you use are volumes of water you can find how much it stretches for each added 100 g of water. You can then use it to measure other masses. You need a plastic (or strong paper) cup. Punch small holes in it in four places around the top and use string to make loops. This will let you link it to a long rubber band. (If you cannot get a long one, loop several shorter ones together in a chain.)
2 Tie one or more rubber bands to the cup and also to a
 support. It should look like this.
The starting length of the rubber band, when not stretched by additional masses, is shown in the diagram. Measure it.
Complete a table showing the mass added and the amount that the rubber band has stretched from its starting length.

| Mass added | 0 | 100 g | 200 g | 300 g |
| :--- | :--- | :--- | :--- | :--- |
| Stretch (cm) | 0 |  |  |  |

Show students how to plot these points on a graph. They should form a straight line.

## 3 Walking and running - same start

Take your class outdoors. Measure a distance of about 50 metres, marking each 5 metres. One learner should start at the beginning of the track and walk steadily. One other person (maybe teacher) uses a watch to count seconds. The other learners record the position the walker has reached after each second.
Use the results to make a table. For example, if the walk speed is 2 metres per second:

| Number of seconds | 0 | 1 | 2 | 3 | 4 | algebra |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Distance walked | 0 | 2 | 4 | 6 | 8 | $2 t$ |

Show how to draw a graph to show these results.
Distance (metres)
The sloping line is a graph with a gradient of 2. If we use $d$ for the number of metres, and $t$ for the number of seconds, then $d=2 t$.


You may then repeat this activity with either faster walking or running. The speed will be greater, say 5 metres per second, and the formula will be $d=5 t$.

## 4 Walking and running - head start

You can extend this activity with a person walking, say at $2 \mathrm{~m} / \mathrm{s}$, but starting further along the track. If the walker starts at 3 metres, then the results for walking at 2 ms will be this.

| Number of seconds | 0 | 1 | 2 | 3 | 4 | algebra |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Distance walked | 3 | 5 | 7 | 9 | 11 | $2 t+3$ |

Here is the graph.
Distance
(metres)
The sloping line is a graph with a gradient of 2 . If we use $d$ for the number of metres, and $t$ for the number of seconds, then $d=2 t+3$.


## Spreadsheets

## Algebraic expressions

This shows a table, a set of strips and squares and a graph, all for the same formula. You need to use the information to decide the two numbers in the formula $y=\square x+\square$. There are four tabs, depending on whether the two numbers are positive or negative.
Travel (demo)
This shows distances travelled at different speeds in 'real life'. The numbers for each speed (walk, run, ride, drive, fly) can be change for one hour, and the graphs show the distancetime graph, with the speed showing up as the slope of the line. Students can be asked to find the formula for each case.
Graph guessing
A graph is shown, and it has a gradient ( $m$ ) and intercept (c). students guess the formula (using the gradient and intercept) and as they do so the line they have chosen is shown in red. There are four tabs: according to whether the gradient or intercept is positive or negative.

## c Solving linear equations

1 Using strips and squares - type 1
Type 1 means Expression = number. The method here uses the balancing, or 'do the same to both sides' approach. It is helped by using strips and squares.
If at all possible, make photocopies of the page xx in this book that has grey strips and squares. The copies will already be white on the back. Then these may be used by groups of students to solve the problems by creating the diagrams that appear below.

b $2 n+4=9$


$3 n=1$, so $n=\frac{1}{3}$

## d $3 n+4=9$



$$
3 n=5 . \text { so } n=1 \frac{2}{3}
$$

e $2 n-4=5$


Add 4 to both sides to make
$2 n=9$, so $n=4.5$
Here are a set of equations your students might solve while using the strips and squares.
$2 n+1=5 \quad 2 n+2=8 \quad 2 n+3=8 \quad 3 n+2=8 \quad 3 n+3=15$
$2 n-1=5 \quad 3 n-3=7 \quad 3 n-1=5 \quad 3 n-3=6 \quad 4 n-1=5$

It is good for learners to check the answers by substituting the proposed solution into the left side.

## Solving equations - games with playing cards

Shuffle the cards. On a piece of paper, draw three spaces for the cards. Take two cards.
Place the bigger number on the $\square$ space and the smaller number on the $\Delta$.
Find $n$, the missing number that makes the equation true.
The answer may be a fraction.
Multiply only
Multiply before adding
Multiply before subtracting

$$
\begin{aligned}
& 2 n=\square \\
& 2 n+\Delta=\square \\
& 2 n-\Delta=\square
\end{aligned}
$$

$$
\square=5 n
$$

$$
\Delta+5 n=\square
$$

$$
\square-5 n=\Delta
$$

$$
(n-\Delta) \times 5=
$$

2 Using strips and squares - type 2
Type 2 means two equal expressions. This time it will be necessary to add or subtract the strips representing the unknown. Again, both demonstrating and letting students work with strips and squares in groups will help them understand quickly.


Subtract $n$ and 4 from both sides.| This leaves $2 n=2$, so $n=1$.
2 a $4-3 n=n+6$


Add $3 n$ to both sides.
Subtract 4 from both sjdes.
$0=4 n+2$


Subtract 2 from both sides
This gives ${ }^{-2}=4 \mathrm{n}$.
Divide by 4 to get $n=-0.5$.
Here are a set of equations your students might solve with the strips and squares.
$3 n=n+10$
$4 n=n+15$
$3 n+4=n+10$
$4 n+3=n+12$
$5 n-2=2 n+4$
$6 n-5=n-1$

It is good for learners to check the answers by substituting the proposed solution into both sides.

## Spreadsheets

## Solving equations

This randomly chooses numbers for five different types of equations:

$$
a x+b=c, a x+b=c x, a x+b=c x+d, a(x+b)=c \text { and } a(x+b)=c(x+d) .
$$

The student is asked to choose the operation to do to both sides, and to type the numbers that result from doing what they have chosen.

## d Change the subject of a formula - with backtracking

Literal equations (that is, formulas with letters) are complex. Because the numbers are now replaced by letters, we can no longer use hands-on material or draw pictures. The method below (Backtracking) is often useful, and can be easier for equations with numbers as well. However it only works for Type 1 equations (only numbers on the right side). This is normally the case for a formula.
1 Backtracking with numbers
In this method we work backwards by 'undoing' the steps that were used to get the right side of the equation. It is essential that students understand the order of operations.
Example: $2 n-3=6$
Start with $n$, the multiply by 2 . Lastly take away 3 to get 6 .
Working backwards, start with 6 , but add 3 , to get 9 ; then divide by 2 to get $4 \frac{1}{2}$.
We can show it more briefly this way.

| Forwards | $n$ | $\times 2 \rightarrow$ | $2 n$ | $-3 \rightarrow$ | $2 n-3$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Backwards | 4.5 | $\leftarrow \div 2$ | 9 | $\leftarrow+3$ | 6 |

2 Backtracking with letters
Example: $a n-b=c$
Start with $n$, the multiply by $a$. Lastly take away $b$ to get $c$.
Working backwards, start with $c$, but add $b$, to get $b+c$; then divide by $a$ to get $\frac{b+c}{a}$.
We can show it more briefly this way.

| Forwards | $n$ | $x a \rightarrow$ | $a \mathrm{n}$ | $-b \rightarrow$ | $a n-b$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Backwards | $\frac{b+c}{a}$ | $\leftarrow \div a$ | $b+c$ | $\leftarrow+b$ | $c$ |

## Spreadsheets

## Backtracking

This generates a set of equations when you type n and ENTER into cell B2. Then you must type a and ENTER (for 'answer'). Type the answers into the yellow cells. If the colour changes to green you are correct!

## e Expand linear expressions

1 Tables, strips and squares
Since algebraic expressions are generalisations of number patterns, we should be able to use numbers to show each pattern in a table.
There is a very useful geometric model for algebraic expressions, using strips and squares, that helps many learners to understand expanding and factorising.
(We will also use these methods in Form 2 to assist the learning of quadratic expressions expanding binomials and factorising trinomials.)
Explanation for teacher:
The primary syllabus makes no reference to the Distributive law, which deals with the property of numbers that $a(b+c)=a b+a c$.
However they do teach multiplication of a two-digit number by a one-digit number.
For example
$4 \times 15=4 \times(10+5)=4 \times 10+4 \times 5$.
It is normally set out like this, and this clearly shows that students are familiar with this number property.

| 15 |  |
| :--- | :--- |
| $\underline{\times 4}$ |  |
| 20 | $4 \times 5$ |
| $\underline{40}$ | $\underline{4 \times 10}$ |
| $\underline{60}$ | $\underline{4 \times 15}$ |

It will help their understanding if you link this 'new knowledge' with what they already know.
In this approach we are using the 'strips and squares' pictures to make pictures to show a factorised expression. The width of the rectangle will be the common factor, and the length will be the other factor.
Here are some examples. The tables below them demonstrate that the factorized expression has the same value as the two terms that are added.
$2 n+2=2(n+1) \quad 3 n+3=3(n+1) \quad 2 n-2=2(n-1)$


| $n \quad 01234$ | n 01234 | n 01234 | n 01234 |
| :---: | :---: | :---: | :---: |
| $2 n+2246810$ | $3 n+3 \quad 3691215$ | 2n-2 -20246 | $3 n-3$-3 0369 |
| $2(n+1) 246810$ | $3(\mathrm{n}+1) 3691215$ | $2(\mathrm{n}-1)-20246$ | $3(\mathrm{n}-1)-30369$ |

Here are a set of expansions your students might do with the strips and squares.
$3(n+1) \quad 2(n+3) \quad 4(n+1) \quad 2(n-1) \quad 3(n-2) \quad 3(2 n+1)$
Each should be checked with a table as in the examples above.

## Spreadsheets

Expanding brackets
This allows the user to change the numbers for $a, b$ or $c$ in the expression $a(b n+c)$ and see the number pattern for different values of n from 0 to 10 .

## f Factorise linear expressions

1 Tables, strips and squares
Explanation for teacher:
Factorising, or adding brackets, is the exact opposite of expanding, or removing brackets.
To do so we need to look for a common factor of the terms. This is illustrated with the number pattern

$$
30+12=6 \times 5+6 \times 2=6 \times(5+2)
$$

We can make a rectangle from the strips and squares representing the expression. When expanding we started with a rectangle - when factorising we go back to the rectangle.
Make sure they look for the common factor.
If at all possible, make photocopies of the page $x x$ in this book that has grey strips and squares. The copies will already be white on the back. Then these may be used by groups of students to solve the problems by creating the diagrams that appear below.

$3 n+18=3(n+6)$
$3 n-18=3(n-6)$
If $n=1,3+18=3 \times 7=21$


If $n=1,3-18=3 x^{-5}=-15$


Here are a set of expansions your students might do with the strips and squares.
$2 n+6$
$3 n+6$
$2 n-6$
$3 n-6 \quad 4 n+6$
$6 n-4$

Each should be checked with a table as in the examples above.
The last two demonstrate that you must find the highest common factor for the width of the rectangle. The answers are $2(2 n+3)$ and $2(3 n-2)$.

## Spreadsheets

Common factor
Two numbers are presented; they have a common factor greater than 1. They many have several common factors. Your task is to type the highest common factor. The graph may help; it may also assist your understanding.
Factorising (adding brackets)
This lists the factors of the expression. For most factors the numbers are not all integers. The one using only integer values are shown in green.

## g Algebraic fractions

I do not know a hands-on method to help with this topic. However many students can get the idea by revising fraction operations with numbers first.

$$
\text { Example: } \frac{2 c}{a}+\frac{c}{3 a b}
$$

Replace the letters by small numbers, different from the numbers in the problem. Let us choose $a=4, b=5$ and $c=6$. Do not multiply the numbers. So the expression becomes:

$$
\frac{2 \times 6}{4}+\frac{6}{3 \times 4 \times 5}=\frac{2 \times 6 \times 3 \times 5+6}{3 \times 4 \times 5}
$$

Now replace the numbers by letters, to get $\frac{2 \times c \times 3 \times b+c}{3 \times a \times b}=\frac{6 b c+c}{3 a b}$.

## C Geometry

For many students, geometry is an enjoyable break from numbers and algebra, which make up most of mathematics at this level. However geometry is full of ideas, and each idea has its own words - in English.
One of the good things about geometry is that it is easy to do simple physical activities to help students grasp the ideas. Then all we need to do is to make sure that they also learn the new English words to go with them. We do that by asking questions, and expecting the correct words in their answers.
There has been quite a lot of geometry done in primary school. You should expect to find that students know quite a lot, so build on their previous knowledge.

## a Angles and constructions

## 1 Arm turning

Students stand and hold out their two arms horizontally to for an angle (with a point in their body as the vertex). Use this to show that it is not the length of the arms that makes the size of the angles but the amount of turning between the arms. Students make right-angles A, B and C $\left(90^{\circ}\right)$, straight angles $\left(180^{\circ}\right)$, acute, obtuse and reflex angles.
The same activity can be done with one arm vertically above another.


## 2 Parallel lines and transversals

The important ideas regarding parallel lines and transversals are a) which angles are equal, and b) which add to $180^{\circ}$, not their names.
Students rule a line that crosses parallel lines on a lined page.
They cut out a slip of paper that fits onto one angle and

- show all the other angles that are equal to the cut-out angle;
- show which angles add to the cut-out to make $180^{\circ}$.


## Spreadsheets

Angles as turns
This draws an angle size and one arm of the angle is rotated to fit.

## b Triangles and constructions

## 1 Sorting triangles

Equilateral triangles are a subset of isosceles triangles
It is all a matter of definition. The equilateral triangles are a special case of isosceles triangles, because they have not only two sides equal, but also the third side is equal to the other two.


## 2 Outdoor constructions

Groups go outdoors with instructions below.

## Three sides known (SSS)

- Draw a base of 2 m , with other sides 2 m and 1 m . (This is isosceles.)
- Draw a base of 2 m , with other sides 2 m and 2 m . (This is equilateral.)
Two sides and the angle between them (SAS)
- Draw a base of 2 m , with an angle $45^{\circ}$ (fold a sheet of paper at one corner) and the other side 2 m . (This is isosceles and acute.)

- Draw a base of 2 m , with an angle $90^{\circ}$ (use a sheet of paper) and the other side 2 m . (This is isosceles and right-angled.)
- Draw a base of 2 m , with an angle $135^{\circ}$ (use two sheets of paper $90^{\circ}+45^{\circ}$ ) and the other side 2 m . (This is isosceles and obtuse-angled.)


## Two angles and side between them (ASA)

- Draw a base of 2 m , with one angle $45^{\circ}$ (fold a sheet of paper at one corner) and the other angle $45^{\circ}$. (This is isosceles and rightangled.)
- Draw a base of 2 m , with one angle $90^{\circ}$ (estimate if needed) and the other angle $45^{\circ}$. (This is also isosceles and right-angled.)



## 3 Angle sum activities

## Paper tearing

On a sheet of paper, each student draws a different large triangle; try to get them to avoid the lazy use of the right-angles. They colour the three angles inside the triangle. They then cut out the triangle.
After tearing off the corners they can fit the three angles together to form $180^{\circ}$.

## Paper folding

On a sheet of paper, each student draws a different large triangle; try to get them to avoid the lazy use of the right-angles. They then cut out the triangle. This time they fold each angle to a point on the base directly below the top vertex. The three angles add to $180^{\circ}$.


## Outdoor triangles with loops of string

Give three students a loop of string (about 10 m long). They stand holding the string at one point each to form a triangle. Other students use an outdoor protractor to measure the sizes of the three angles, which should add to $180^{\circ}$.

## 4 Triangle area

Halving a rectangle
For any triangle you may always draw a rectangle around it. The triangle will have half the area of the rectangle.


## Paper cutting and rearranging

For any triangle you may always find the midpoints of two sides, draw the two vertical lines, cut off the two triangles and move them into the blue positions shown. The result of a rectangle of the same height but half the base of the triangle. So the triangle area is half of base x height.


## Outdoor measurement activity

Outdoors, students need to draw a large triangle. Each side should be at least 1 metre long. Then choose one side as the base and measure it. With the tape measure (or string) make the vertical height (at right-angles to the base), and measure it. Calculate the area. Then choose another side as the base, and repeat the measurements and calculation. The illustration below shows what to do if the triangle is obtuse: extend the base until it is 'under' the top vertex.


## Spreadsheets

## Sorting triangles

A triangle is shown. It is either acute-angled, right-angled or obtuse-angled; the student chooses one. It is either isosceles or scalene; the student chooses one. Delete answers for a new triangle. This would make a good demonstration.

## Constructing triangles

Four tabs allow the user to create triangles by typing three side lengths (SSS), two angles and the side between them (ASA), two side lengths and the angle between them (SAS) and two side lengths and the angle not between them (ASS). The first three will always create one triangle, but the fourth (ASS) can sometimes create two triangles, or even no triangle at all! This would make a good demonstration. There is a page of investigations, called Explore, which you may print.
Angle sum for triangles
Using the angles that are equal for parallel lines (see below), students show that the sum of the angles $(A+B+C)$ is $180^{\circ}$. the shape of the triangle may be changed, using $F 9$.


## Triangle areas

A triangle is drawn on a graph grid. there are three types of problem: 1) given base and height, find area, 2) given base and area, find height and 3) given height and area, find base.

## c Polygons

At Form 1 level, this will include an introduction to quadrilaterals (or a reminder, since they are in the primary syllabus). It will include many other types of polygons, both regular (with same side lengths and angles - and a lot of symmetry) and irregular.
1 Making quadrilaterals from A4 paper

## A Square

To fold a rectangle of plain paper into a square, follow the diagrams.


## B Rhombus (diamond)

Fold the rectangle in half in both directions and then cut off the corners at the same time. Open out.


## C Parallelogram

Cut a triangle from one end, and use it to cut exactly the same triangle from the other end.


## D Trapezium

Cut a triangle from one end of the rectangle.


## E Isosceles trapezium

Fold the rectangle in half. Then cut a triangle from both thicknesses at once. Open out.


## F Kite

Fold the rectangle in half. Then cut off both corners. Open out.


## 2 Tangrams

Tangrams are a Chinese puzzle involving seven pieces made by cutting up a square. Put letters on them so we can say which to use to solve the puzzles.


Trace this shape or draw it larger and cut out the seven pieces. Then use them to make these shapes.

- Rectangle with d, e, f - Rectangle with d, e, g - Parallelogram with d, e, f
- Trapezium with d, e, c
- Rectangle with c, d, e, g
- Parallelogram with $\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{g}$
- Trapezium with $\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{g}$
- Square with c, d, e, f, g
- Rectangle with $\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{f}, \mathrm{g}$
- Parallelogram with c, d, e, f, g

Use all seven to make

- a triangle
- a rectangle
- a trapezium
- a parallelogram
- a square
- a hexagon.

Here are the answers:

- Rectangle with d,e,f - Rectangle with d,e,g|
- Parallelogram with $d, e, f$

- Trapezium with d, e, c
- Rectangle with $\mathbf{c}, \mathbf{d}, \mathrm{e}, \mathrm{g}$
- Parallelogram with $\mathrm{c}, \mathrm{d}, \mathrm{e}, \mathrm{g}$

- Square with c, d, e, f, g
- Trapezium with $\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{g}$

- Parallelogram with $c, d, e, f, g$

- Rectangle with all seven pieces

- Rectangle with c, d, e, f, g

- Triangle with all seven pieces

- Trapezium with all seven pieces

Hexagon with all seven pieces

- Parallelogram with all seven pieces



## 3 Angle sum activities

## Quadrilateral

If you cut out any quadrilateral, and tear off each the corners, they will fit together to make a complete $360^{\circ}$.
Any polygon
Draw the polygon, and, from one vertex, draw lines to each of the others. Count the triangles formed; there will be 2 less than the number of sides. each triangle has $180^{\circ}$, so the angle sum is $180(n-2)^{\circ}$.

## Spreadsheets

Angle sum in polygons
For polygons of sides 3 to 6 , this will draw the polygon and its diagonals from one vertex.
The user then completes the table to deduce the formula 180(n-2).

## d Line reflections

Symmetry is not only the basic of attractive patterns but a good way to understand more about shapes. Show some practical applications, on the Malawian flag, in letters and numbers, crafts, fabrics and other things.

## 1 Letters and numbers

These have line symmetry.

## Vertical: AHIMOTUVWXY80 <br> Horizontal: BCDEHIOX380

## 2 Mirror game

Two people stand and pretend that there is a large vertical mirror between the, so each is a reflection image of the other. They then move and try to keep the reflection going.

## Spreadsheets

Flips
This deals with mirror symmetry of three shapes, a triangle $\Delta$, letter P or letter d (choose one). There are six mirror lines you may use, three each horizontally and vertically. There are two tabs, one for students to explore, and one to test their understanding.

## e Circles

## 1 Circumference walk

Two students are involved. One acts as the centre of the circle and holds a string or rope. The other takes a number of normal steps (say 5 ) to reach a point on the circumference of a circle around that centre. The students all guess how many normal steps it will take for the walker to go completely round the circle. Then the walker does this, keeping the string tight. It will be found that the circumference is a bit over 6 times the radius ( $2 \pi=6.28$ ), so for 5 steps the correct answer is about 31 or 32.


## 2 Measuring diameters and circumferences of circles

Students measure the diameters and the circumferences of as many circles as they can find. To measure, they use a length of string to stretch across the diameter and to wrap around the outside of the circle. They measure the length of the string needed using a ruler. They put the results into a table. Also use multiplication to complete the last line.

| Diameter $(\mathrm{cm})$ |
| :--- |
| Circumference $(\mathrm{cm})$ |
| Diameter $\times 3.14(\mathrm{~cm})$ |

## 3 Count squares for area

On a sheet of centimetre grid paper, draw a circle of radius 10 cm . Students count the squares to estimate the area. The answer will be close to $314 \mathrm{~cm}^{2}$, which is $\pi r^{2}$.
(To be more accurate, draw a quarter circle with radius 20 cm , multiplying the answer by 4.)


## Spreadsheets

## Circumference

When a circle (e.g. bicycle wheel) rolls along turning exactly once it rolls a distance of its circumference. This spreadsheet shown this. Use F9 to make it move. You may change the radius.
Circle area
This draws circles with different radii, and shades the number of squares with area more than 1.

## D Statistics

There is a lot of statistics in the primary syllabus. You may find that students already are familiar with many ideas.

## 1 Collect personal data

You will want to use personal (but not private) data to illustrate the types of graphs and averages. Have students measure their heights and their hands-spans; to that they can add their ages and number of siblings (brothers and sisters).
Ages and numbers of siblings make good column graphs, and it is easy to find averages.
Height and hands-spans make good grouped frequency tables (for example each 5 cm range of height), or each 5 mm for handspan. The graph of such tables is called a histogram.
2 Mean, median and mode
The mean is the common average.
The median is the middle one when they are arranged in order. (If there is an even number of numbers, the median is halfway between the middle two.
The mode is the most common - the one with the highest frequency. There might be more than one mode. If all the numbers are different, there is no mode.
Class activity:
Choose seven students who clearly differ in height.
Arrange them in order. Find the median for seven (the middle one).


Then let one of them sit down.
The new median is the average of the middle two heights.


## Spreadsheets

Mean vs median
The user may enter up to 20 numbers for data. The values are plotted (in a line graph), and the mean and the median are calculated and also plotted. Students can clearly see that one or more values much higher than the rest will move the mean (average) higher than the median, but not change the middle point (median) at all.

## 4 Hands-on methods in Form 2

## Introduction

## Matching to the syllabus

The present Form 2 syllabus presents topics in a disconnected way. This book shows the natural links between topics, to make the ideas easier to understand. Not every topic is covered with hands-on activities.

- Number patterns: page 52.
- Sequences: page 52.
- Simplify algebraic expressions: page 52.
- Pythagoras' theorem: page 72.
- Congruency in triangles: page 69.
- Quadrilaterals and their properties: page 69.
- Prove theorems about parallelograms: page 69.
- Surface area and volume of cylinders and pyramids: page 73.
- Simultaneous linear equations: page 55.
- Direct proportion: page 43.
- Inverse proportion: page 47.
- Mixtures: page 48.
- Indices and logarithms, standard form: page 49-51.
- Similar triangles: page 45.
- Inequalities: page 54.
- Distance-time graphs: page 52.
- Quadratic equations: page 60-66.
- Geometric constructions: page 68.
- Circles related to triangles: page 68.
- Probability: page 74.


## A Number

## a Estimating square roots

The square root of a number is that number that multiplies by itself to make the number. That is easy if it is a 'perfect square' whole number. (The square root of 16 is 4 .)
Non-perfect squares (whole numbers or decimals over 1)
This is a hands-on activity involving the use of a calculator. each group may be asked to use multiplication only on their calculator to find the square root of a different number, which is not a perfect square. They will experience the frustration that no answer is exact!
For example, to estimate the square root of 5 , we need to find a number that multiplies by itself to make 5 . Clearly it is a decimal, between $2(2 \times 2=4)$ and $3(3 \times 3-9)$ and closer to 2 .
Try 2.2: $2.2 \times 2.2=4.84$. So 2.2 is too small.
Try 2.3: $2.3 \times 2.3=5.29$. So 2.3 is too big.
Try 2.25 : $2.25 \times 2.25=5.0625$. So 2.225 is too closer, but still too big..
Try 2.23: $2.23 \times 2.23=4.9729$. So 2.23 is a little too small.
Try 2.235: $2.235 \times 2.235=4.995225$. So 2.2235 is a tiny bit too small.
Try 2.237: $2.237 \times 2.237=5.004169$. So 2.2235 is a tiny bit too big.
Try 2.236: $2.236 \times 2.236=4.999696$. So 2.2236 is a tiny bit too small.
and so on.

This may be the first example your students have met of the process called 'iteration' in maths. Iteration is the process of getting closer and closer to an answer by small steps. The concept of an irrational number (a non-repeating decimal)
The process described above for the square root of 5 illustrates the ideas:

- The answer is a decimal that can never be exact. No matter how many decimal places you use, it will always be a little too small or a little too big.
- It cannot be exact because the 'last digit' must be $1,2,3,4,5,6,7,8$ or 9 . And when you multiply any number ending in one of these the answer must end in 1 ( $1 \times 1$ or $9 \times 9$ ), 4 ( $2 \times 2$ or $8 \times 8$ ), 9 ( $3 \times 3$ or $7 \times 7$ ), 6 ( $4 \times 4$ or $6 \times 6$ ) or $5(5 \times 5$ ).


## Square roots of decimals under 1

This idea can be illustrated with numbers that have exact square roots.
Ask them to find the square root by multiplication only.
Examples: the square root of 0.09 is 0.3 ; the square root of 0.25 is 0.5 ; the square root of 0.49 is 0.7 ; the square root of 0.0004 is 0.02 ; the square root of 0.0016 is 0.04 .

The general conclusion: the square root of a number less than 1 is larger than the number! The explanation relates to indices: for example $0.01 \times 0.01=10^{-2} \times 10^{-2}=10^{-4}=0.0001$. This will be useful when learning about indices and logarithms later.

## Spreadsheets

Finding square roots
This does the iterative process for any number you type. You just choose the nearest product that is just under (or equal to) the number you want; then move to the next decimal place, and so on.
One tab uses numbers 2 to 99 and the other 0.2 to 0.99 .

## b Ratios and scale factors

$A$ ratio compares two quantities ( $A$ and $B$ ) by stating how many times bigger $A$ is than $B$. This may be a whole number or a mixed number.
A whole number example: K20 is 5 times K4. We sometimes say this as $5: 1$ ("5 to 1 ").
A mixed number example: $K 10$ is $2 \frac{1}{2}$ times $K 4$. We sometimes say this as $5: 2$ (" 5 to 2 "). If the A is smaller than B the ratio states the fraction that A is of B .
A fraction example: K2 is $\frac{1}{2}$ of K 4 (or $1: 2$ ).

## 1 Aspect ratios of rectangles

The 'aspect ratio' of a rectangle compares its width to its height.
A doorway will have a fraction as an aspect ratio, since width is smaller than height.
A TV screen will have a mixed number as an aspect ratio, since width is more than height.
Students should find many rectangles around the classroom, measure them in centimetres, and divide the width by the height to calculate the aspect ratios.

## 2 Orange cordial

When making a cordial with concentrated juice and water, the 'concentration' of the drink depends on the ratio of water to concentrate. Instructions often look like this: 'Mix 4 parts of water to one part of concentrate'. This is the ratio $4: 1$. Students should not that this makes the concentrate one-fifth ( $\left(\frac{1}{5}\right)$ of the total mixture.

## Spreadsheets

## Estimating ratios

This shows a green line and a red line. You ae asked to estimate the ratio of green to red. Dots along the lines help in the comparison.

## Orange cordial

The spreadsheet states how much concentrate (orange) and how much water (blue) is in two different jugs; the pictures are only approximate. Students are asked to say which has the higher concentration of orange, or if they are equal.

## c Direct proportion

Direct proportion refers to situations where one quantity always stays the same ratio of another. A proportion statement says that two ratios are equal.

## 1 Distance $=$ speed $x$ time

If you travel at a constant speed, the distance you go depends on the time taken. This is an example of direct proportion.
Students can go outside and measure their walking (or running) speeds by finding the time taken to travel a fixed distance. They can complete a table and find a formula; for example:

| time in seconds $(t)$ | 0 | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| distance in metres $(d)$ | 0 | 5 | 10 | 15 | 20 |

The ratio of distance to time taken is called speed. The formula is $d=5 t$.
Once they know the speed they can work out the distance for any time, or the time taken to go any distance, at that speed. They can draw a graph using the formula, or the table, and use the graph to find distance (given time) or time (given distance).

## 2 Cost = unit price $x$ quantity

When buying goods the more you buy the more it costs. The ratio of the cost to the quantity is called the unit price, such as K60 per litre.
This relates closely to algebra, as we can create a table and a formula, and draw a graph.

| Volume in litres $(x)$ | 0 | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Cost in $K(y)$ | 0 | 60 | 120 | 180 | 240 |

The formula is $y=60 x$, and the graph will be a straight line through the origin, with a gradient of 60 .
We notice that when the volume doubles the cost also doubles.
This allows us to calculate the cost for any volumes and also the volume for any cost.

## 3 Use the ideas of equivalent fractions

We can treat a proportion just like equivalent fractions.
For example, suppose 4 kilograms of flour costs K300. what would be the cost of 6 kilograms? We call the unknown cost $c$.
The ratio of cost to kg is $\frac{300}{4}$. This is equal to the ratio $\frac{c}{6}$. So $\frac{300}{4}=\frac{c}{6}$.
The simplest method here is to divide both 300 and 4 by the common factor 2 . So $\frac{150}{2}=\frac{c}{6}$.
Now the denominator 6 is 3 times the denominator 2, so c $=3 \times 150=450$. Answer K450.

## 4 Unitary method

Many people find this method easier to understand. It is called 'unitary' because you first find 1 of the unknown, before you multiply.
Example 1: You can buy 4 kg of something for K60. How much can you buy for K90?
We set it out this way, with the unknown second.
K60 for 4 kg
divide by 60 K 1 for $4 \div 60 \mathrm{~kg}$
multiply by 90 K 90 for $4 \div 60 \times 90=6 \mathrm{~kg}$.
Example 2: You can buy 4 kg of something for K60. What is the cost of 5 kg ?
We set it out this way, with the unknown second.
4 kg for K60
divide by $4 \quad 1 \mathrm{~kg}$ for $\mathrm{K} 60 \div 4$
multiply by 5 K 5 for $60 \div 4 \times 5=\mathrm{K} 75$.

## 5 Using bottle tops for increasing or decreasing using a ratio

Sometimes we use ratios to talk about how much we want to increase, or decrease, a quantity. Students may use bottle tops to represent the numbers involved.
For example, the class size at a particular school is 24, and increases in the ratio $5: 4$. What is the new class size?
One way to think of what this means is this:
Here are the 24 in 6 groups of 4.
For every 4 students in the original 24 , there are now 5 . Now each 4 increases to become 5 .

So now there are 6 groups of $5=30$.
Using the approach of equivalent fractions: $\frac{5}{4}=\frac{n}{24}$, so $n=6 \times 5=30$.
Decreasing in a ratio works the same way, with each group size being reduced.

## Card games: increases and decreases using ratio

Use the pack of cards for the numbers 1 to 6 and 10 only. (These are factors of 60.) Shuffle the pack.
Each player imagines they have K60.
Each player gets two cards. They make a ratio (larger to smaller) to increase the K60, and work out the answer.
Now take the answer and swap the numbers in the ratio, so that the number now decreases the amount. The answer should be K60.

## Spreadsheets

## Proportional cordial

For the idea, see ‘Orange cordial' above.
In this spreadsheet the amounts of concentrate and water are given for one mixture.
For another jug a different amount of concentrate is given and we are asked to work out the amount of water that will give the two jugs the same strength of cordial.
Travel
The sheet 'Travel (demo)' lets you change the speeds for walking, running, riding driving and flying. This changes the table and the graphs.

The other tabs are designed to help you and your students to learn more about how to use a spreadsheet program.
The sheet 'Table’ gives instructions for making your own spreadsheet with table and graph. The sheet 'Start+steps' uses a more efficient way to create the table.
The sheet 'Rule' uses a formula to create the table.

## d Similar triangles

Similar figures are enlargements (or reductions) of the size of a shape. It is a picture approach to the idea of enlarging (or reducing) using a ratio. The basic idea is the scale factor for the enlargement (or reduction). This is the ratio of any length in the enlarged (or reduced) shape compared to the length in the original.

## 1 Rubber band for scale factors

Use two cards to create two-digit numbers (use $\mathrm{A}=1$, and pictures $=0$ ), e.g. 23. Create two of them (e.g. 23 and 59), and estimate the ratio of one to the other, and vice versa. A useful way to visualise this is to imagine the numbers as the lengths of lines on a number line.


This is similar to the spreadsheet 'Estimating ratios' above.

## 2 Enlarging using two rubber bands

You need two rubber bands looped together. Hold one end fixed firmly in place. The far end will stretch and always stay twice as far as the knot where they join. Make a small drawing (e.g. a rectangle) and use this to enlarge it, by moving the knot over the drawing and using a pencil at the moving end of the rubber bands.


## 3 Heights using shadows

During the day the length of the shadows of trees, buildings etc changes. But at any one time, the length of the shadow depends on the vertical height. (This assumes that the shadow is on horizontal land.)


Students measure the vertical heights of many outdoor objects (including themselves) and measure the lengths of horizontal shadows. Put these into a table and find a formula. Use the formula to calculate the shadow length for other heights, or the height for other shadow lengths. (This last step may be used to find the heights of trees, etc.)

## 3 Mirror on the floor

Similar triangles may be created indoors using reflection from a small mirror placed on the floor. Outdoors you may use the reflections in a puddle of water!


## 4 River width

Students could test out this method for finding the width across a river, without going over the water.


We need to find the length $d$ (metres). We create two right-angled triangles using two trees or marked points. We measure lengths $b$ and $c$. We find the length $a$ so that the two trees are in line of sight; this makes the triangles 'similar' - they have the same angles.
Then we use the lengths in equivalent fractions like this: $\frac{d}{a}=\frac{b}{c}$.
Note that, if $b=c$, then the triangles are congruent, and $d=a$.

## 5 Vertical protractor and scale drawing

To find the height of a tree or tall building, measure three things from a point some distance away on level ground.

- Measure the angle from the horizontal. $\left(A^{\circ}\right)$.

Use a protractor vertically (with thread a weight) to measure the angle of elevation.

- Measure the height of your eye. (h m)
- Measure the angle between the horizontal and the top of the tree or building. (m)


With the measurements made, students create a scale drawing of the situation using a large piece of paper, a ruler and a protractor. (The vertical object must be represented by a vertical line on unknown length, but its height will be found from where the angle meets that line.)

## 5 Map scales

If you are able to show maps to students they can find the scales and use them to calculate distances. This also is a use of proportion and similar figures

## Spreadsheets

Size change
You create a shape using coordinates, and type a scale factor to see the result. Scale factors may be any kind of number, including negative.

## River width

This allows you to use the scroll bars to change the value of $a$ and $c$ in the diagram above. The value of $b$ is always 10 metres. You then calculate the width of the river using proportion.

## e Inverse proportion

In direct proportion the ratio of the two amounts $\left(\frac{y}{x}\right)$ is always the same.
In inverse proportion it is the product of the two amounts ( $x y$ ) that is always the same.
1 Sharing for constant product
Imagine you have found K600 and, being a good person, you want to share it with your friends.
On your own, you get K600. If two people share you get K300. If three people share you get K200. Students can make a table to show how much each person gets when K600 is shared by different numbers of people. (there will be some numbers, like 7, where the sharing cannot be exact, but you can work it out to the nearest whole number.

| Numbers of shares | 1 | 2 | 3 | 4 | 5 | 6 | 10 | 12 | 15 | 20 | 25 | 30 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Size of share | 600 | 300 | 200 | 150 | 120 | 100 | 60 | 50 | 40 | 30 | 24 | 20 |

Although you work out the size of the share by dividing, the product is always 600 .

## 2 Rectangles with constant area

Ask students to draw as many rectangles with an area of 60 as they can. Even using whole numbers there are many. The product of width and height is always the same.

| width | 1 | 2 | 3 | 4 | 5 | 6 | 10 | 12 | 15 | 20 | 30 | 60 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| height | 60 | 30 | 20 | 15 | 12 | 10 | 6 | 5 | 4 | 3 | 2 | 1 |

## Spreadsheets

## Constant-area rectangles

You just type the area you want, and many (but clearly not all) rectangles are drawn. They all have one corner at the origin, so their top right corners form a curve, called a hyperbola.

## f Relative density

1 Paper boat
Each group will need a bucket partly full of water, and of course their paper boat. You also need a bucket of sand for the whole class. This will enable them to carefully add sand and see the boat get lower into the water until it eventually fills with water and sinks to the bottom. At least use it as an interesting demonstration. Let us try an experiment with a paper boat and stones. Use an A4 sheet of paper.
Fold it on these lines so it folds up into an open box, with the flaps inside the box. Glue or staple the flaps. Float the boat on water.
Now carefully pour some sand into it so that it sinks deeper into the water, and eventually sinks.


## Spreadsheets

## Floating

This actually shows a lump of 'whatever' floating (or not floating) in 'whatever liquid'. You are able to change the density of the liquid (or even a gas) and see it float or sink. The densities of many materials $(\mathrm{g} / \mathrm{mL})$ are given.

## g Mixtures

Use students and a see-saw (or pretend one) to demonstrate this basic idea.
When a person sits on a see-saw their ability to balance the person on the other side of the balance point depends on two things: their weight, and how far from the balance point they sit. Here is an example of two people sitting at different distances from the balance point, but balancing because they are different weights.


Let us imagine that two people are balanced on this see-saw. But the lighter person is three times as far from the balance point as the heavier person. This shows that the heavier person is three times the weight of the lighter person.
Mixtures work this way too. Materials that are more dense need smaller amounts to balance the materials that are less dense, which need larger amounts.
We call this way of calculating 'using weighted averages'.

## Spreadsheets

## Mixtures

This spreadsheet has two tabs.

- The first is to find density of a mixture, when you know the density and the volume of two materials that are mixed.
- Multiply the volumes by the density of each to find the mass that each contributes to the mixture.
- Then divide the total mass by the total volume to find the density of the mixture.
- The second is to find fractions of total that are each component of the mixture.


## h Indices

1 Fold paper strips for powers of 2
Lead students through this activity.
Use sheets of A4 paper. Fold it length-wise into 8 strips and cut the strips.
each time we fold a paper strip we double the number of sections (between the folds).
The original strip $1=2^{0}$


Two folds four sections $4=2^{2}$


Three folds eight sections $8=2^{3}$


| Number of folds | 0 | 1 | 2 | 3 | 4 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Number of sections | 1 | 2 | 4 | 8 | 16 |
| Powers of 2 | $2^{0}$ | $2^{1}$ | $2^{2}$ | $2^{3}$ | $2^{4}$ |

So this is a 'real' way of showing the meaning of positive powers of 2 .
Notice that it also shows that $2^{0}=1$.
Negative powers of 2
To think about negative powers of 2 we need to think about the length of each section when folding.
Start with a length of 1 , then fold but look at the fraction of the original length.
The original strip $1=2^{0}$ $\square$

One fold, length $\frac{1}{2}=2^{-1}$


Two folds length $\frac{1}{4}=2^{-2}$


Three folds, length $\frac{1}{8}=2^{-3}$


| Number of folds | 0 | 1 | 2 | 3 | 4 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Length of sections | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ |
| Powers of 2 | $2^{0}$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ |

So this is a 'real' way of showing the meaning of negative powers of 2 .
Notice that it also shows that $2^{0}=1$.

## Puzzle - Tower of Hanoi

You need three pieces of paper of different sizes (B Big, M Middle, S Small).
On a sheet of paper, mark three rings where these may be put.
Start with the pieces of paper on ring a - biggest $B$ on the bottom, then middle $M$, and smallest S on top: BMS.


Your challenge is to move these pieces, one at a time, to finish in the same order SMB in a different ring. But there is one rule: you must NEVER put a larger piece on top of a smaller piece.
It can be done, and takes 7 moves. $B$ has 1 move, $M$ has 2 moves, and $S$ has 4 moves. If you use four different pieces of paper, it takes 15 moves ( $1+2+4+8$ ).

## Spreadsheets

## Paper folding

This shows on screen the activities above, but is no substitute for doing the real thing. Index laws with numbers
This randomly produces two expressions that include indices, and these are multiplied or divide or both. When right you get a tick. For convenience, the indices are not superscripted.

## h Standard form

## 1 Digit slider

A number in standard form has one digit left of the decimal point. It is best to think of the digits in the number moving (not the decimal point). To get from

### 135.79

to

$$
1.3579
$$

the digits have to move right by two places. So we multiply by 100 (or $10^{2}$ ) to compensate. To go the other way, just move the left digit left and include zeros if needed.
So $3.1 \times 10^{3}$ becomes 3100 because the 3 moves to the thousands place.
Students create strips of paper from A4. They write the digits spread out on the strip of paper.

$$
\begin{array}{|lllllll|}
\hline 1 & 3 & 5 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

They put a pencil between the digits where the decimal point is supposed to be.

$$
\begin{array}{lllllll|}
\hline 1 & 3 & 5 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Now you can move the digits to the right so that the number shown is between 1 and 10 .

$$
\begin{array}{|l|llllll|}
\hline 1 & 3 & 5 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

You move the digits 6 places so we multiply by $10^{6}$.

$$
1,350,000=1.35 \times 10^{6}
$$

## Standard form for small numbers

Of course this is just the same as for large numbers, but the digits move left to produce standard form, so we must multiply by a negative power of 10 .
From this will come the understanding that:

- multiplying by $10^{-1}$ is dividing by $10^{1}$
- multiplying by $10^{-2}$ is dividing by $10^{2}$
and so on.
Write the digits spread out on a sheet of paper.

| 0 | 0 | 0 | 0 | 1 | 3 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |

Then put a pencil between the digits where the decimal point is supposed to be.

| 0 | 0 | 0 | 0 | 1 | 3 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |

Now you can move the digitsto the left so that the number shown is between 1 and 10 .

| 0 | 0 | 0 | 0 | 1 | 3 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |

You move the digits 4 places so we multiply by $10^{-4}$.

$$
0.000135=1.35 \times 10^{-4}
$$

## i Logarithms

## 1 Use powers of 2

The logarithm is just the exponent, or index number.
We have this table from the paper strip folding. We have just extended it.

| Logarithm (folds) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of sections | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| Powers of 2 | $2^{0}$ | $2^{1}$ | $2^{2}$ | $2^{3}$ | $2^{4}$ | $2^{5}$ | $2^{6}$ | $2^{7}$ | $2^{8}$ | $2^{9}$ |

Show students how this can be used to demonstrate the index laws.

- Add logarithms to multiply the numbers

Example: $8 \times 64=512$. This is just $2^{3} \times 2^{6}=2^{9}$. We can add the indices (logarithms) to multiply the numbers.

- Add logarithms to divide the numbers

Example: $128 \div 32=4$. This is just $2^{7} \div 2^{5}=2^{2}$. We can subtract the indices (logarithms) to divide the numbers.

## 2 Use powers of 10

| Power of ten | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of sections | 1 | 10 | 100 | 1000 | 10000 | 100000 | 1000000 | 10000000 |
| Powers of 2 | $2^{0}$ | $2^{1}$ | $2^{2}$ | $2^{3}$ | $2^{4}$ | $2^{5}$ | $2^{6}$ | $2^{7}$ |

Show students how this can be used to demonstrate the index laws.

- Add logarithms (indices) to multiply the numbers

Example: $100 \times 10000=1000000$. This is just $10^{2} \times 10^{4}=10^{6}$. We can add the indices (logarithms) to multiply the numbers.

- Add logarithms (indices) to divide the numbers

Example: $100000 \div 100=1000$. This is just $10^{5} \div 10^{2}=10^{3}$. We can subtract the indices (logarithms) to divide the numbers.

## Spreadsheets

## Using logarithms

The spreadsheet uses a graph of $y=10^{x}$. The numbers are on the vertical $(y)$ axis, and the indices (logarithms) on the horizontal ( $x$ ) axis.
Instructions are given for using the graph to find logarithms, and then to add them to multiply the numbers or subtract them to divide the numbers.
Teachers who remember logarithm tables will recall the process from generations past.

## B Algebra

The activities in Form 2 follow on from learners enjoying the algebra activities in Form 1. However if you are short of time, and if your class has not done the Form 1 Algebra activities, you could do only those for Form 2.

## a Travel graphs

1 Measure speeds outdoors ( $\mathrm{m} / \mathrm{s} \rightarrow \mathrm{km} / \mathrm{h}$ )
You need a measured distance (e.g. $10,20,50$ or 100 m ) and the ability to measure time accurately. Since speed is distance divided by time you may need a calculator (or good estimation skills) to work it out.
Speed units depend on the distance and time units used. The common units for vehicles is $\mathrm{km} / \mathrm{h}$, but for walking or running we use $\mathrm{m} / \mathrm{s}$.
If you are asked about converting between these units, then you can follow this procedure.
$1 \mathrm{~m} / \mathrm{s}=60$ metres $/$ minute $=3,600$ metres $/$ hour
But $3,600 \mathrm{~m}$ is the same as 3.6 km . so $1 \mathrm{~m} / \mathrm{s}$ is $3.6 \mathrm{~km} / \mathrm{h}$.
To convert $\mathrm{m} / \mathrm{s}$ to $\mathrm{km} / \mathrm{h}$. multiply by 3.6 . To convert $1 \mathrm{~km} / \mathrm{h}$ to $\mathrm{m} / \mathrm{s}$. divide by 3.6.

## Spreadsheets

Distance-time-speed
This give two numbers for distance or time or speed and asks you to work out the third.
The graph might be helpful.

## b Linear number patterns and terms of a sequence

## 1 Surrounding activities

Each group needs a collection of 30 bottle-tops. Put one upside-down and put others around it, like this. Then add another upside down and surround them. Do it again.


Learners complete this table. (Here the answers are given.)

| Number upside-down | 1 | 2 | 3 | algebra |
| :--- | :--- | :--- | :--- | :--- |
| Number around them | 8 | 10 | 12 | $2 n+6$ |

They find the formula: start at 8 and add 2 each time.
It goes up by $2 s$, so the formula is $2 n+6$.
We have both a pattern with shapes and a number sequence: $8,10,12, \ldots$. Use your bottle tops to continue this pattern by adding more.
Here are more examples, with answers supplied.
A Two rows upside-down.

$B$ Three rows upside-down.

| width of rows | 1 | 2 | 3 | 4 | algebra |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Around them | 12 | 14 | 16 | 18 | $2 n+10$ |

曲


C Squares. Use $n$ for the side length of the squares.


## Card game - Creating and guessing a formula

One player chooses two cards without showing the other player. The first player uses them in this formula.

$$
a=\square n+\Delta
$$

to create a number pattern using the numbers $0,1,2,3,4,5$. The first player shows the pattern to the other player.
The second player has to say the numbers on the two cards, and also which of the two formulas is being used.
For example, the pattern $3,7,11,15,19,23$ comes from $\square=4$ and $\Delta=3$. This is because the formula used to create the pattern is $a=4 n+3$.

## 2 Crossing the river (if not done earlier)

'Crossing the river' is a logic puzzle that leads to a simple formula. Tell the story of a family who come to a small river and have to cross using a small raft or boat. The raft can only hold one or two children, or one adult.
Build up the formula from the activity, by changing the number of adults who need to cross. Then students find the rule (or formula) and plot the graph. The multiplier in the rule (4) is the gradient of the graph. It shows the number of trips needed for each extra adult.
The rule is $t=4 n+1$. The extra 1 is the first (or the last) trip that gets the children over. So the graph starts at the point 1 . (This means there is 1 trip needed before any adults can get over.)
You must do the activity in groups. It will be enjoyable.
The table and rule are:

| number of adults, n | 0 | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: | :---: |
| number of trips, t | 1 | 5 | 9 | 13 |



The graph is linear, but we should not join the dots. This is because adults come in whole numbers, not fractions!
The gradient of the graph shows the number of trips needed for each extra adult. The 1 shows the first (or last) trip.

## Spreadsheets

Gardens (rectangles, squares)
This has diagrams of all the 'surrounding' problems. They are presented as gardens with paving stones around them. The paving stones can either be squares or dominoes (two squares side by side). Enter the numbers into the table and type the numbers for the formula.

## Sequences

This allows you to enter the start number, and the step size. The table shows the sequence, and the graph shows it as well. There is another tab where the student types the first term and the step size, from the table and graph.

## c Solving linear inequalities

It is the graphing of solutions that is new in this topic.
1 'Stand on number line' game
Each group needs to prepare a number line, say from -4 to +4 . The numbers are far enough apart that people can stand on two numbers side-by-side.
The group members stand near the line with each person having a number.
When an inequality is stated, each person decides whether or not their number is part of the solution. If their number is part of the solution they step onto that number position on the line, thus forming the graph using people.
Example 1: for $n+3<5$, numbers less than 2 will be part of the solution: 1, $0,-1,-2,-3,-4$. Example 2: for $n+3 \leq 5$, numbers less than or equal to 2 will be part of the solution: 2, 1, 0 , $-1,-2,-3$, and -4 .

## Card game - Solving inequalities

Shuffle the cards. Choose two cards with different numbers.
On a piece of paper, draw three spaces for the cards. Place the smaller number on each of the $\Delta$ spaces and the larger on $\square$.
Find $n$, the missing whole numbers that make the inequality true.

| Adding: $n+\Delta<\square$ | $n+\Delta \leq \square$ | $n+\Delta>\square$ | $n+\Delta \geq \square$ |
| :--- | :--- | :--- | :--- | :--- |
| Subtracting: $n-\square<\Delta$ | $n-\square \leq \Delta$ | $\square-n>\Delta$ | $\square-n \geq \Delta$ |

## Spreadsheets

## Inequality graphs

The graph for an inequality (showing integer values only) is shown. You are asked to:
a choose $<, \leq,>$ or $\geq$ to make it correct for the number shown
b given the correct symbol, type the correct number, or
c given only the graph, choose the symbol and then type the number.

## d Simultaneous linear equations

This topic brings together the ideas of linear relationships graphing and solution of equations.

## 1 Graphs of travel (table, graph and substitution)

Travel graphs are graphs of distance against time.
One person runs down a road at $2 \mathrm{~m} / \mathrm{s}$.
Another person starts at the same time, but 20 metres ahead. This person walks at $1 \mathrm{~m} / \mathrm{s}$. Does the runner catch up to the walker? If so, after how many seconds, and now many metres along the road?
It is good to take the class outside and actually try to do this. You might arrange for small groups to follow instructions and do it for themselves.
Then take them inside, complete a table and draw a graph.

| Time from start (seconds) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Walker distance (metres) | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| Runner distance (metres) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |

The table shows that the walker and the runner will meet after 6 seconds, at 12 metres from the start.
Now draw the graph using the table


The graph shows that the walker and the runner will meet after 6 seconds, at 12 metres from the start.

Finally use the table or graph to work out the formula for each line:
walker $d=t+6$, runner $d=2 t$.
These are equal when $t+6=2 t$. The solution is $t=6$. When $t=6, d=12$.
This shows that the table, the graph and the algebra all give the same answer.
(The method used here is 'substitution'. We have substituted $2 t$ for $d$ in $d=t+6$.
this should not be confused with the substitution of numbers for letters in formulas.)

## 2 Heads and legs puzzle (table and elimination)

Puzzle: A family keeps some hens and cows. There are 6 heads, and 16 legs.
How many hens and how many cows?
Learners will use trial and error to work this out. They could use a table to list the possible combinations.

| number of hens | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| number of cows | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| Number of legs | 24 | 22 | 20 | 18 | 16 | 14 | 12 |

The table shows that the answer is 4 hens, 2 cows for 16 legs.
Here is the graph.


The table shows that the answer is 4 hens (so 2 cows) for 16 legs.
The two variables are number of hens ( h ) and number of cows ( c ).
The two relationships involved are: $h+c=6$ (for heads) and $2 h+4 c=16$ (for legs).
Divide the legs relationship by 2 . So $h+2 c=8$

$$
\text { and } \mathrm{h}+\mathrm{c}=6
$$

Subtract the second from the first: $\quad c=2$ (two cows) so there are 4 hens.
Check 2 cows ( 8 legs) and 4 hens ( 8 legs) making 16 legs in total.

## 3 Buns and drinks (elimination)

Puzzle: A shop sells buns and drinks. What is the cost of a bun and the cost of a drink?


Ask learners to explain to each other and to you HOW they solved this problem. There are many ways to solve this problem. You might discuss these with the class. Here are some; you might find others.
(It is unlikely that your students will use algebra to explain their method, but once they have explained it, you need to show them how to use algebra to represent their thinking.)

## Method A

If two buns and two drinks cost K440, then one of each cost K220.
This means that in the second row, the bun and one drink cost K220. So the two drinks left cost K80, making them K40 each. Therefore the bun is K180. Here it is in algebra.

$$
\begin{aligned}
2 B+2 D & =440 \quad \text { Divide both sides by } 2 \\
B+D & =220
\end{aligned}
$$

In the second equation replace $B+D$ by 220 .

$$
\begin{aligned}
B+3 D & =300 \text { becomes } \\
220+2 D & =300 \\
\text { So } 2 D & =80 \text { and } D=40 .
\end{aligned}
$$

Since $B+D=220$, we have $B+40=220$.
So

$$
\mathrm{B}=180 .
$$

## Method B

The first row has an extra bun, and the second row has an extra drink. So the price difference between bun and drink is K140. This means the bun is worth K140 plus the cost of a drink.
In the first row you can replace each bun by K140 and one drink - so we have four drinks and K280 worth K440. This means K160 is the cost of the four drinks, making them K40 each. The bun must be K180. Here it is in algebta.

$$
\begin{aligned}
2 B+2 B & =440 \\
B+3 D & =300 .
\end{aligned}
$$

Subtract the second equation from the first.

$$
B-D=140,
$$

therefore $B=D+140$. so $2 B=2 D+280$.
In the first equation, replace 2 B by $2 \mathrm{D}+280$.

$$
2 D+(2 D+280)=440
$$

Subtract 280 from both sides

$$
4 D+280=440
$$

So $4 \mathrm{D}=160$, divide by 4 to get $\mathrm{D}=40$

## Method C

If we double the second row, we have two buns and six drinks costing K600. The difference between the first and the second rows is now four drinks, for K160. So each drink is K40 In the second row the drinks cost K120, making the bun K180. Here it is in algebra.
Since $B+3 D=300$, double both sides

$$
2 B+6 D=600
$$

But 2B+2D=440
Subtract this from the equation above it:

$$
\begin{aligned}
4 D & =160 \\
D & =40
\end{aligned}
$$

Put $D=40$ into $\quad B+3 D=300$

$$
\begin{aligned}
B+3 \times 40 & =300 \\
B+120 & =300 \\
B & =180
\end{aligned}
$$

## More bun and drink problems:

a The cost of 3 buns and 2 drinks is K190, and the cost of 4 buns and one drink is K170. Find the cost of each.
b The difference between the cost of three buns and a drink is K50. The cost of two buns and three drinks is K290. Find the cost of each.
c Three buns cost the same as two drinks. Two buns and three drinks together cost K195. Find the cost of each.

## Spreadsheets

Graphical solutions
Type the numbers for the two equations in the form $a x+b y=c$, and $d x+e y=f$.
The lines are graphed and the intersection is named.
You can find examples where there are no solutions, when the graphs are parallel, having the same gradient.

## e Quadratic functions and graphs

This will be the first time students have met the parabola. It is common in everyday life.

## 1 Parabola from lines

Ask students to draw two 20 cm lines at right angles - one vertical and one horizontal.
Along each line mark each centimetre. Number the marks from the intersection $0,1,2,3$ up to 20 .
Join numbers that add to 21 . So 1 on the vertical line joins to 20 on the horizontal line, and so on. The result looks like this. It is called a parabola.


## 2 Ball toss parabola

The most common example of parabolas are the paths of balls, or water from hoses. Sketch a parabola on the board and have students try to make a ball follow the path of the parabola as they throw it.


Parabolas were originally studied as the paths of shells fired in battle.

## 3 Kangaroo puzzle

Ask three boys and three girls to step out to the front of the class. They will sit on 7 chairs, leaving a gap in the centre. They will pretend to be kangaroos (or rogs if you prefer). The boys must pass the girls, and the girls pass the boys, using these rules for the possible moves in the game:

- 1 step forward into a vacant spot, and
- 2 jump forward over only one of the opposite sex.

It is possible, and takes 15 moves.
Students in groups work on completing the table below for different numbers of kangaroos (frogs) at each end, but always the same number of each. Always have only one empty chair in the centre at the start and end.
Here are the answers.

| Number at each end | 0 | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Number of moves | 0 | 3 | 8 | 15 | 24 |

There are three different forms of the same formula for the number of moves.

$$
m=k^{2}+2 k, m=k(k+2) \text { and } m=(k+1)^{2}-1 .
$$

It is not necessary to find the formulas in order to graph the points in the table and see the parabola appear.

## Spreadsheets

Projectiles
This shows the path of a ball tossed at a speed and an angle. It follows a parabola.
Choose a speed and an angle and see how far it goes.
Now change the angle only and find another angle that goes the same distance.
What angle makes it go the furthest?
What angle makes it go the highest?
On another tab is 'Target' where you must choose the speed and angle to hit the target.

## f Solving quadratic equations using tables and graphs

1 Areas of rectangles with the same perimeter
Using grid paper students draw many examples of rectangles with perimeters of 28.
There are many: $1 \times 13,2 \times 12,3 \times 11,4 \times 10,5 \times 9,6 \times 8,7 \times 7$, $8 \times 6,9 \times 5,10 \times 4,11 \times 3,12 \times 2,13 \times 1$.
They all fit the relationship $w(14-w)$, where $w$ is the width.


Students complete a table of the width related to the area of the rectangle.

| width (w) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| :---: | :---: | :---: | :---: | :---: | :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| height | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| area (a) | 13 | 26 | 33 | 40 | 45 | 48 | 49 | 48 | 45 | 40 | 33 | 24 | 13 |

Discuss with students how they may use the table to obtain two solutions to these equations: $w(14-w)=13, w(14-w)=26, w(14-w)=33, w(14-w)=40, w(14-w)=45$ and $w(14-w)=48$.
Point out that there is only one solution for the equation $w(14-w)=49$. It is 7 , and is the square. A square is a rectangle that has the biggest area for the same perimeter.
Using the table students draw the graph of this relationship between width and area.
Areas of rectangles with perimeter 28


Discuss with students how they may use the graph to obtain two approximate solutions to these equations: $w(14-w)=5, w(14-w)=10, w(14-w)=15, w(14-w)=20, w(14-w)=$ $25, w(14-w)=30$ and $w(14-w)=35$.
Point out that there is no solution for the equation $w(14-w)=50$. The area cannot get that big while the perimeter stays at 28!

## Spreadsheets

Quadratic equations
A quadratic equation is presented, in the form $n \times n+6=5 \times n$. All the ns must be the same number. It will have one or two whole number solutions; only equations with one or two whole number solutions from 0 to 9 are used (and there are 100 possible equations). Students can test any possible solution by typing it. When they have found the one or two solutions they type them both to complete the problem.

## g Expanding binomials

This section is about different expressions that always give the same answer; these are called identities. The approach suggested here is to show examples of the number pattern represented by the identity. Then the algebra can be understood for what it is: a general statement describing a number pattern.

1 Long multiplications and general case
TYPE $1(10+a) x(10+b)$
We start by making the link to multiplication of two 2-digit numbers.
The multiplication $12 \times 13$ has four parts:
$10 \times 10,10 \times 2,10 \times 3$ and $2 \times 3$.
Using base 10 material it looks like this.
You can see it adds up to $100+20+30+6=100+50+6=156$.


If we are multiplying $16 \times 19$ we can use the same pattern:
$100+60+90+54=100+150+54=304$
Student should try this for other multiplications of the form $(10+a) \times(10+b)$.
2 Other multiplications ( $12 \times 7$ ) and general case TYPE $2(10+a) x(10-b)$.
If we follow the same pattern, the multiplication $12 \times 7$ has four parts: $(a=2, b=3)$ $10 \times 10,10 \times 2,10 \times-3$ and $2 \times-3$. Using base 10 material it looks like this. The coloured parts are subtracted.
 You can see it adds up to $100+20-30-6=100-10-6=84$.

$(10+a) \times(10-b)$.
If we are multiplying $16 \times 9$ we can use the same pattern: $(a=6, b=1)$
$100+60-10-6=100+50-6=144$
Student should try this for other multiplications of the form $(10+a) \times(10-b)$.
3 Other multiplications $(8 \times 7)$ and general case $(10-a) \times(10-b)$
TYPE 3 ( $\mathbf{1 0}-a) x(10-b)$.
If we follow the same pattern, the multiplication
$8 \times 7$ has four parts: $(a=2, b=3)$
$10 \times 10,10 \times-2,10 \times-3$ and $-2 \times-3$.
Using base 10 material it looks like this.
The coloured parts are subtracted.
You can see it adds up to
$100-20-30+6=100-50+6=56$.
$(10-a) \times(10-b)$.
If we are multiplying $6 \times 9$ we can use the same pattern: ( $a=4, b=1$ )
$100-40-10+4=100-50+4=54$


Student should try this for other multiplications
of the form $(10+a) \times(10-b)$.
4 Make a rectangle with strips and squares, and check with a table
Clearly we have been using examples in which $x=10$. Now we move to a general expression. Student may use the template of strips and squares. Some are coloured (for negative) and some are not.
TYPE $1(x+a)(x+b)=x^{2}+(a+b) x+a b$
For example $(x+2)(x+3)=x^{2}+(2+3) x+6$. $=x^{2}+5 x+6$.
Check in a table

| $x$ | $(x+2)(x+3)$ | $x^{2}+5 x+6$. |
| :--- | :--- | :--- |
| 0 | $2 \times 3=6$ | $0+0+6=6$ |
| 1 | $3 \times 4=12$ | $1+5+6=12$ |
| 2 | $4 \times 5=20$ | $4+10+6=20$ |
| 3 | $5 \times 6=30$ | $9+15+6=30$ |
| 10 | $12 \times 13=156$ | $100+50+6=156$ |



TYPE $2(x+a)(x-b)=x^{2}+(a-b) x-a b$
For example $(x+2)(x-3)=x^{2}+(2-3) x-6$. $=x^{2}-x-6$.
Check in a table

| $x$ | $(x+2)(x-3)$ | $x^{2}-x-6$. |
| :--- | :--- | :--- |
| 0 | $2 \times-3=-6$ | $0-0-6=-6$ |
| 1 | $3 \times-2=-6$ | $1-1-6=-6$ |
| 2 | $4 \times-1=-4$ | $4-2-6=-4$ |
| 3 | $5 \times 0=30$ | $9-3-6=0$ |
| 10 | $12 \times 7=84$ | $100-10-6=84$ |



TYPE $3(x-a)(x-b)=x^{2}-(a+b) x+a b$
For example $(x-2)(x-3)=x^{2}-(2+3) x+6$.

$$
=x^{2}-5 x+6 \text {. }
$$

Check in a table

| $x$ | $(x-2)(x-3)$ | $x^{2}-5 x+6$. |
| :--- | :--- | :--- |
| 0 | $-2 \times-3=6$ | $0-0+6=6$ |
| 1 | $-1 \times-2=2$ | $1-5+6=6$ |
| 2 | $0 \times-1=0$ | $4-10+6=-4$ |
| 3 | $1 \times 0=0$ | $9-15+6=0$ |
| 10 | $8 \times 7=56$ | $100-50+6=56$ |



## 5 Perfect squares

Students will be familiar with the numbers $1,4,9,16,25$ etc.
TYPE $1(10+a) \times(10+a)=10^{2}+20 a+a^{2}$
Examples with numbers
$11 \times 11=(10+1) \times(10+1)=100+20+1=121$
$13 \times 13=(10+3) \times(10+3)=100+60+9=169$
$15 \times 15=(10+5) \times(10+5)=100+100+25=225$
The general pattern looks like this.
$(x+a)(x+a)=(x+a)^{2}=x^{2}+2 a x+a^{2}$
This is an example of TYPE 1 where $a=b$.
The diagram shows $(x+3)^{2}=x^{2}+6 x+9$
The width of the rectangle is $(x+3)$.
The height of the rectangle is $(x+3)$.


The area is $x^{2}+6 x+9$

TYPE $3(10-a) \times(10-a)=10^{2}-20 a+a^{2}$
Examples with numbers
$9 \times 9=(10-1) \times(10-1)=100-20+1=81$
$7 \times 7=(10-3) \times(10-3)=100-60+9=49$
$5 \times 5=(10-5) \times(10-5)=100-100+25=25$
The general pattern looks like this.
$(x-a)(x-a)=(x-a)^{2}=x^{2}-2 a x+a^{2}$
This is an example of TYPE 3 where $a=b$.
The diagram shows $(x-3)^{2}=x^{2}-6 x+9$
The width of the rectangle is $(x-3)$.
The height of the rectangle is $(x-3)$.


The area is $x^{2}-6 x+9$

6 Sum x difference (= difference of two squares)
$(10+a) \times(10-a)=10^{2}-a^{2}$
Examples with numbers
$11 \times 9=(10+1) \times(10-1)=100-1=99$
$13 \times 7=(10+3) \times(10-3)=100-9=91$
$15 \times 5=(10+5) \times(10-5)=100-25=75$
The general pattern looks like this.
Sum $x$ difference $=$ difference of squares

$$
(x+a)(x-a)=x^{2}-a^{2}
$$



This is an example of TYPE 2 where $a=b$.
The diagram shows $(x+3)(x-3)=x^{2}-6 x+9$
The width of the rectangle is $(x+3)$.
The height of the rectangle is $(x-3)$.
The area is $x^{2}-9$.

## Spreadsheets

## Expand binomials

This draw the pictures of the rectangles illustrated above. You may change the values of $a, b$ and $x$. the three types are in different tabs at the bottom of the screen.
Difference of two squares
This shows the number pattern 'sum x difference $=$ difference of two squares' in two ways.

## h Factorising quadratic expressions

1 Quadratic number patterns involving factorising
If we put different values in place of $n$ in a quadratic expression, we can sometimes find factors. Here is an example. (Students can complete the table.)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $n^{2}+5 n+6$ | 6 | 12 | 20 | 30 | 42 | 56 |
| factors? | $2 \times 3$ | $3 \times 4$ | $5 \times 6$ |  |  |  |

Write a general expression for the factors of $n^{2}+5 n+6$.
(The factors are $(x+2)(x+3)$ in each case.)
Another example:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $n^{2}-1$ | -1 | 0 | 3 | 8 | 15 | 24 |
| factors? | $-1 \times 1$ | $0 \times 2$ | $1 \times 3$ | $2 \times 4$ | $3 \times 5$ | $4 \times 6$ |

Write a general expression for the factors of $n^{2}-1$.
(The factors are $(x-1)(x+1)$ in each case.)
Another example:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $n^{2}-2 n-8$ | -8 | -9 | -8 | -5 | 0 | 7 |
| factors? | $-4 \times 2$ | $-3 \times 3$ | $-2 \times 4$ | $-1 \times 5$ | $0 \times 6$ | $1 \times 7$ |

Write a general expression for the factors of $n^{2}-2 n-8$.
(The factors are $(x-4)(x+2)$ in each case.)

The factors are not always generalisable.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $n^{2}+1$ | 1 | 2 | 5 | 10 | 17 | 26 |
| factors? | $1 \times 1$ | $1 \times 1$ | $1 \times 5$ | $2 \times 5$ | $1 \times 17$ | $2 \times 13$ |

The expression $n^{2}+1$ has no factors.

## 2 Make a rectangle area with strips and squares

## Type 1: All terms positive

The process of finding factors is just the reverse of expanding. We know that $(x+2)(x+3)=x^{2}+5 x+6$
So we can work backwards.
We make a rectangle and look at its width and height.
This is straightforward when all terms are positive.
A special case is the perfect square:

$x^{2}+2 \mathrm{a} x+a^{2}=(x+\mathrm{a})^{2}$

## Type 2: Last term negative

If the number term is negative, then to the square we must add positive in one direction and add negative in the other.
For example $\quad x^{2}-x-6=x^{2}+2 x-3 x-6$.

$$
=(x+2)(x-3)
$$

The middle term is split into two parts, one positive and one negative.
A special case is the difference of squares:
$x^{2}+2 \mathrm{a} x+a^{2}=(x-\mathrm{a})(x+\mathrm{a})$


Type 3: Last term positive. middle term negative
If the number term is positive, then to the square we must add negative in both directions. This results in a positive number term.
For example $x^{2}-5 x+6=x^{2}-2 x-3 x+6$.

$$
=(x-2)(x-3)
$$

The middle term is split into two parts, both negative.
A special case is the perfect square:
$x^{2}-2 \mathrm{a} x+a^{2}=(x-\mathrm{a})^{2}$


## Spreadsheets

## Expand binomials

This draw the pictures of the rectangles illustrated above. You may change the values of $a, b$ and $x$. the three types are in different tabs at the bottom of the screen. You can easily produce the specials cases of perfect squares, and difference of two squares.

## h Solving quadratic equations given factors

1 Diagonals of polygons
The number of diagonals of a polygon (with 3, 4, 5, etc sides) depends on the number of sides. Students sketch the polygons, accurately enough to be able to draw and count the number of diagonals. They complete the table (below).

| $n$ sides | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $d$ diagonals | 0 | 2 | 5 | 9 | 14 | 20 |
| $2 d$ | 0 | 4 | 10 | 18 | 28 | 40 |
| factors? | $3 \times 0$ | $4 \times 1$ | $5 \times 2$ | $6 \times 3$ | $7 \times 4$ | $8 \times 5$ |

So $2 d=n(n-3)$ and $d=\frac{n(n-3)}{2}$
They might reason that each vertex has $(n-3)$ diagonals coming from it, but this counts each diagonal twice.

## Spreadsheets

## Solve a factorised equation

This presents three types of equation: $(x-a)(x-b),(x-a)(x+b)$ and $(x+a)(x+b)$.
This can also produce the perfect squares and sum x difference (difference of two squares). For each a table shows how each expression is calculated for different values of $x$.
The student types the $x$-values that make the expression equal to 0 . The student then types the two numbers needed to make the expanded expressiOon, and sees that this has the same solutions.

## C Geometry and measurement

There has been much geometry in primary school and Form 1. Connect with what they know.

## a Dissection puzzles to review geometric knowledge

Dissecting something means to cut it up, so dissection puzzles are when we cut up and shape and put it together to make a different shape. Here are two simple examples. See the template at the end of this book.
Three-piece puzzles
A square is cut into three pieces as in the diagram. From 2 or 3 pieces, students may make triangles, trapezia, squares, rectangles parallelogram pentagons and irregular quadrilaterals.

## Four-triangle puzzles

From square squares drawn side by side four congruent triangles are
 cut, as in the diagram.
From 2, 3 or 4 triangles they may make triangle, rectangle, parallelogram, kite, quadrilateral, rhombus, arrowhead, pentagon, hexagon and square.


## b Symmetry and properties of a rhombus

The rhombus symmetry makes the constructions easier to understand.
A rhombus has two diagonals that bisect each other at right angles. So each is the perpendicular bisector of the other.
In so doing each diagonal bisects two angles of the rhombus, so it is an angle bisector.
Start with a sheet of A4 paper. Fold it in half and then again at right angles to the first fold. This makes a smaller rectangle. Fold this along the diagonal that does not pass through the first two folds, and use this last fold as a guide to cutting away half the rectangle. What is left is a rhombus, with its lines of symmetry shown.


## 1 Make and measure your own rhombus

- Mark two points to be the opposite vertices of a rhombus. Join these points with a line.
- Fold the paper so that one point sits on top of the other.
- Crease the paper and open out. (The crease is the perpendicular bisector of the line joining the two points.)
- Mark a third rhombus point on the crease line, not on the line joining the first two points.
- Fold the paper along the line joining the first two points.
- Mark the point where the third point lands on the first crease. This is the fourth point of the rhombus. Join the vertices.




Students will be able to measure the sides (they should all be equal in length) and the angles (opposite angles should be equal).

## Spreadsheets

Complete the rhombus
This shows you three vertices of a rhombus. You are asked to type the coordinates of the fourth point. If you are right the rhombus is drawn. You can see the vertical and horizontal lines of symmetry.

## c Constructions

## 1 Paper folding

Perpendicular bisector:

- Mark two points to be the opposite vertices of a rhombus.
- Fold the paper so that one point sits on top of the other.
- Crease the paper. The crease is the perpendicular bisector of the line joining the two points.
Angle bisector:
- Draw the arms of an angle of any size. Fold the paper so that one arm sits on top of the other, and crease the paper. The crease is the angle bisector, and should go through the vertex of the angle.



## 2 Outdoors and string for perpendicular bisector

Go outside to a flat space. Mark two points, say 2 to 3 metres apart.
Use a piece of string of about 5 m . Fold it in half, and mark the midpoint.
Two students take the ends of the string to the two points, and another pulls it tight and marks (on the ground) the two positions of the midpoint of the string. Join these two new points: this is the perpendicular bisector. It is clear that we have constructed a rhombus.


## 3 Outdoors and triangle for angle bisector

Mark the vertex V and the two arms of an angle outdoors. (You might find some angle already drawn, such as $90^{\circ}$.) Use a length of string to find two points (A, B) equal distances from the vertex on the arms. From those points find a point $C$ that is the same distance from $A$ and $B$. Join CV; this is the bisector of angle AVB. Again, we have drawn a rhombus.


## Spreadsheets

Perpendicular bisector
This demonstrates the perpendicular bisector for any two points. You tap F9 to move the points around.

## Angle bisector

This shows that any point along the angle bisector is the same distance from the arms of the angle. Tap the scroll bars and use F9.

## d Congruent triangles

Two triangles (or other shapes) are congruent if one can be fitted exactly on top of the other. This means that the instructions lead to identical or congruent triangles.
In this section we let students 'discover' that for certain conditions there is only one triangle that may be drawn. (Note: although there might seem to be two, one might be a reflection of another.) So there are four 'congruency conditions'.
You can only make congruent triangles if you are given:

- three sides of a triangle, (SSS)
- two sides and the angle between them (SAS)
- two angles each side of a base (ASA)
- right angle, hypotenuse and side (RHS)

But if you are given two sides and the angle not between them, there could be two, one or no triangles possible (ASS).

## 1 Outdoor constructions of triangles

See Form 1 Geometry B\#2 on page xx.
The important idea now is that any of the 'congruency' instructions for drawing a triangle can only result in one triangle, or it mirror image. Any such triangle could be placed on top of any other to fit exactly; this is the idea of congruence.

## Spreadsheets

## Constructing triangles

Four tabs allow the user to create triangles by typing three side lengths (SSS), two angles and the side between them (ASA), two side lengths and the angle between them (SAS) and two side lengths and the angle not between them (ASS). The first three will always create one triangle, but the fourth (ASS) can sometimes create two triangles, or even no triangle at all! This would make a good demonstration. There is a page of investigations, called Explore, which you may print.
e Quadrilaterals - properties and areas
1 Tangrams (if not done in Form 1)
See Form 1 Geometry page 37.


## 2 Sort using properties



## Subsets

Firstly look at subsets. Squares are a subset of rectangles; squares are also a subset of parallelograms, of kites, and of trapeziums.

- Squares are subsets of Rectangles, Rhombuses, kites, parallelograms, isosceles trapeziums, trapeziums and quadrilaterals.
- Rectangles are subsets of parallelograms, isosceles trapeziums, trapeziums and quadrilaterals.
- Rhombuses are subsets of kites, parallelograms, trapeziums and quadrilaterals.
- Kites are subsets of quadrilaterals.
- Parallelograms are subsets of isosceles trapeziums, and quadrilaterals.
- Isosceles trapeziums are subsets of trapeziums and quadrilaterals.


## Intersections

The shape must have properties that make it belong to two sets. For example, squares are both rhombuses and rectangles.
Unions
The shapes must have the properties of one other shape, OR another OR both. For example rhombuses, squares and rectangles have four equal sides OR four right angles OR both (in the case of squares).
3 Symmetry of quadrilaterals
Students should make each quadrilateral from a sheet a A4 paper; see instructions on page 36.


Square


Rhombus


Parallelogram Rectangle


Kite


Trapezium

They can fold the shapes over to find the mirror lines (the kite and isosceles trapezium have only one mirror line, but the rectangle and rhombus have two mirror lines - at right angles, and the square has four mirror lines).
The tricky one for some students is the parallelogram. It 'looks symmetrical' and so it is with rotation (half-turn) symmetry. But many students will what to put in a mirror line.

## 4 Diagonals and properties

The two diagonals of quadrilaterals may or may not be equal, cross at right angles or divide each other exactly in half. In each case this relates to the symmetry and the propertes of the shape. Students may explore this for each type of quadrilateral.


## 5 Areas of quadrilaterals

Use this hands-on activity to review how each formula was obtained.
The basic idea is to convert the quadrilateral to a rectangle of the same area.
Parallelogram - move a triangle, so it has the same area are the rectangle.


Rhombus - add two triangle areas


Kite - add two triangle areas


Trapezium - move two triangles to use average width

## Spreadsheets



## Hunting quadrilaterals

Three points of different quadrilaterals are given so that two sides are drawn. The user types the missing vertex, using symmetry ideas, and it is drawn to show you are correct. Sorting quadrilaterals
One of the several types of quadrilaterals is drawn. The student is given a series of questions about parallel sides, equal side lengths and equal angles. By typing Y or N the quadrilateral type is identified.

## Quad areas

This is a demonstration. For each type of quadrilateral its area is related to that of a rectangle. You can vary the shape of the quadrilateral with the F9 key.

## f Theorem of Pythagoras

## 1 Paper drawing and cutting

In this proof students will take the five shapes cut from the two squares on the shorter sides of the triangle, and put them together to form the larger square.
It will work for any shaped right-angle triangle.

- Draw the right-angled triangle. On the shorter side draw a square.
- On the larger side draw a square and use its diagonals to find its midpoint.
- through the midpoint of the larger square, draw a line parallel to the hypotenuse of the right-angled triangle. Then draw another line through the midpoint perpendicular to that.
- Cut out the five pieces. Reassemble them to form a square that fits onto the hypotenuse.



## 2 Pythagoras against a wall

For this each group needs a straight stick one metre long. Obviously a metre ruler is best, but any straight stick of 100 cm will do.

- Lean it against the wall in any position. Mark the height on the wall. take the ruler away.
- Without measuring, calculate what the distance $X$ should be using Pythagoras' theorem.
- Put the rule back at the position you marked, and this time measure to check your calculation.
- Do this several times.


## Spreadsheets

## Squares and triangle



A right-angled triangle is shown and the squares are drawn on each of the three sides. The middle-sized square is split into four parts using the construction in 1 above. In this way it shows that the result applies no matter what shape is the right-angled triangle.

## Pythagoras

This tests for whether or not a triangle is right-angled. It draws the triangle.

## g 3D shapes - properties, surface areas and volumes

The only way that students can really learn about 3D shapes is to make them, from nets. This requires paper, rulers, scissors and glue (or sticky tape). See templates at end of book. 1 Cube nets
Students need a large number of squares of the same size. They will find as many ways as possible to assemble these into nets that fold up into cubes. (There are 11 such nets. They are pictured below.)


## 2 Use a net of an octahedron

An octahedron is a 3D shapes made from 8 equilateral triangles. Here is one net.


## 3 Triangular prism from net - SA and $V$

Student should make the solid from the net, then measure and calculate these.
The surface area is the area of the net.
The volume is the area of the triangle $x$ the length of the rectangles.


## 4 Square pyramid from net - SA and V

Student should make the solid from the net, then measure and calculate these.
The surface area is the area of the net.
The volume is the area of the triangle $x$ the length of the rectangles.


## 5 Sketching 3D shapes

For each of the shapes they construct students should spend time trying to draw them so the 'look right'. This takes time, practice and advice but is important to be able to interpret other people's diagrams later.

## Spreadsheets

## Cube nets

This will show one of many sets of six squares joined along either edges. If it is a cube net, type $Y$, if not type $N$. Delete for a new problem.

## D Probability

## Introduction

Some things in life are certain: death and taxes come to mind. But some things are uncertain. Probability is about predicting the chances that a particular uncertain thing will happen in the future.
There are two ways to look at probability:

- the theoretical way, which is based on the use of 'equally-likely outcomes' and
- the experimental way, which simply records what happens when you repeat the activity many times.
The confusing thing for many students is that the theoretical calculations can suggest that you will get a particular fraction of trials with a particular result (for example in 600 die rolls you will get 100 of each number) - but it often doesn't work that way.


## 1 Dice and cumulative success fractions

Students run an experiment by rolling a die many times, and work out the relative frequencies for each number (the decimal fraction of rolls that give you each number). Results could be entered into a table like this. Different groups or students might focus on different die numbers. Possible results are given.

| Number of rolls | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of 1s so far | 2 | 2 | 5 | 5 | 6 |  |  |  |  |  |
| Fraction of 1s | 0.2 | 0.1 | 0.2 | 0.125 | 0.12 |  |  |  |  |  |

## 2 'Point-up' drawing pins

What is the chance that a drawing pin (thumb tack') will land with its point up?
You need a transparent sealed container containing 10 drawing pins.
Students run an experiment by shaking the container many times, and working out the relative frequencies for point-up after each shake. Each roll will mean 10 trials, because there are 10 pins in the container.
Results could be entered into a table like this. Possible results are given.

| Number of trials | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of ups so far | 4 | 5 | 6 | 12 | 15 | 24 |  |  |  |  |
| Fraction of ups | 0.4 | 0.25 | 0.2 | 0.4 | 0.3 | 0.4 |  |  |  |  |

## Spreadsheets

Drawing pin
This does the experiment with drawing pins. But instead of a mystery number for the probability, you must choose the probably to use.
One experiment (results): This shows the success or failure at each 'toss'. and works out the relative frequency as you go.
One experiment (graph) does the same but graphs the relative frequencies.
Four at once graphs all 100 results, and does it with four separate experiments at once.

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